Multi-weighted Reachability Games

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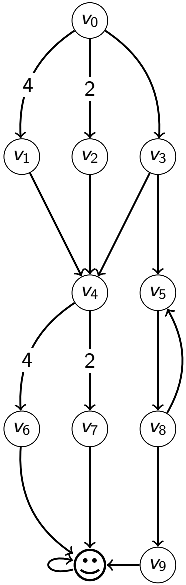
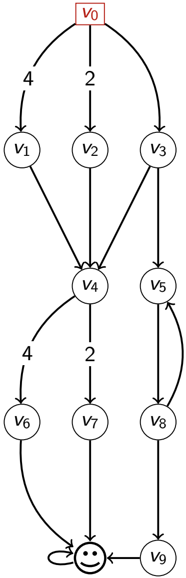
RP'23

Reachability Games



Is it possible to reach ☺ with cost $\leq c$?

Reachability Games

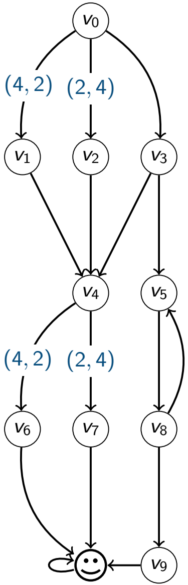


Is it possible to reach ☺ with cost $\leq c$?

↓

Is it possible to reach ☺ whatever the behavior of the **environment** with cost $\leq c$?

Reachability Games



Is it possible to reach ☺ with cost $\leq c$?

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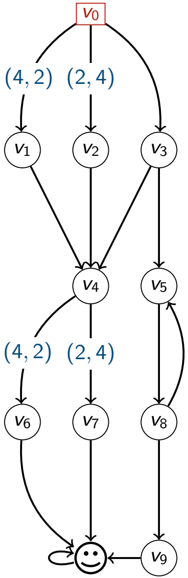
↓ Is it possible to reach ☺ with cost $\leq (c_1, c_2)$?

Reachability Games



Is it possible to reach ☺ with cost $\leq c$?

↓ ↓

Is it possible to reach ☺ whatever the behavior of the **environment** with cost $\leq c$?

Is it possible to reach ☺ with cost $\leq (\mathbf{c}_1, \mathbf{c}_2)$?

↓ ↓

Is it possible to reach ☺ whatever the behavior of the **environment** with cost $\leq (\mathbf{c}_1, \mathbf{c}_2)$?

Two-player Multi-weighted Reachability Games

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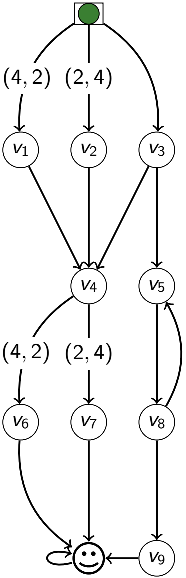
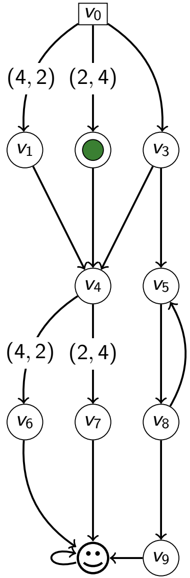


- A d-weighted graph $G = (V, E, (w_i)_{1 \leq i \leq d})$;
- Two players: Player $\bigcirc$ and Player $\square$;
- Turn-based.

A play: v_0

Two-player Multi-weighted Reachability Games



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A play: $v_0 v_2$

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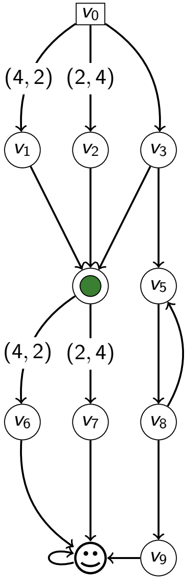

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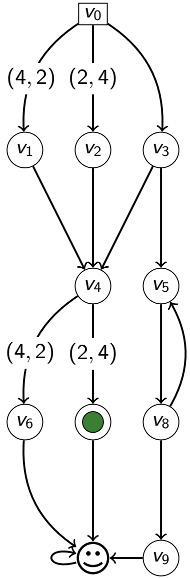
A play: $v_0 v_2 v_4$

Two-player Multi-weighted Reachability Games



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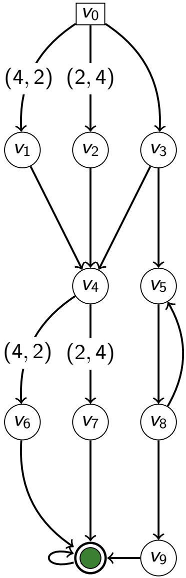
Two-player Multi-weighted Reachability Games



- A d-weighted graph $G = (V, E, (w_i)_{1 \leq i \leq d})$;
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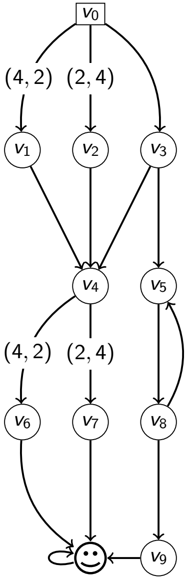
A play: $v_0 v_2 v_4 v_7$☺

Two-player Multi-weighted Reachability Games



- A d-weighted graph $G = (V, E, (w_i)_{1 \leq i \leq d})$;
- Two players: Player $\bigcirc$ and Player $\square$;
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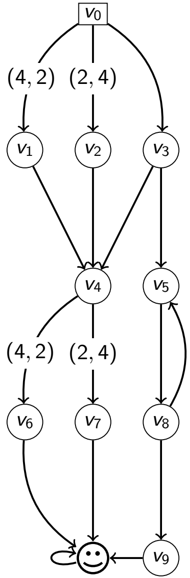
A play: $v_0 v_2 v_4 v_7 ☺^{\omega}$

Two-player Multi-weighted Reachability Games



Quantitative reachability objective

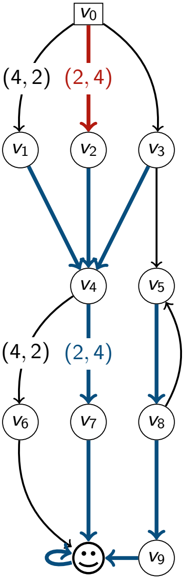
Given a target set $F \subseteq V$, for all plays $\rho = \rho_0\rho_1\ldots$:

$$\mathrm{Cost}_i(\rho) = \begin{cases} \sum_{n=0}^{k-1} w_i(\rho_n, \rho_{n+1}) & \text{if } k \text{ is the least index st.} \rho_k \in F \\ +\infty & \text{otherwise} \end{cases}$$

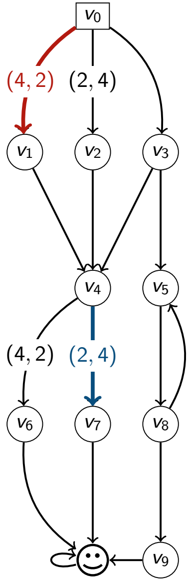
Rem: **same target set** for all dimensions.
Ex:

- $\mathrm{Cost}(v_0 v_3 (v_5 v_8)^\omega) = (\mathrm{Cost}_1(v_0 v_3 (v_5 v_8)^\omega), \mathrm{Cost}_2(v_0 v_3 (v_5 v_8)^\omega)) = (+\infty, +\infty)$
- $\mathrm{Cost}(v_0 v_2 v_4 v_7 ☺^\omega) = (6, 10)$

Two-player Multi-weighted Reachability Games



- A strategy for Player $\bigcirc$: $\sigma_{\bigcirc} : V^* V_{\bigcirc} \longrightarrow V$.
- Given a **strategy profile** $(\sigma_{\bigcirc}, \sigma_{\Box})$ and an initial vertex $v_0 \rightsquigarrow$ only one consistent play $\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_{v_0}$ called the **outcome**.
  Ex: $\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_{v_0} = v_0 v_2 v_4 v_7 ☺^{\omega}$.



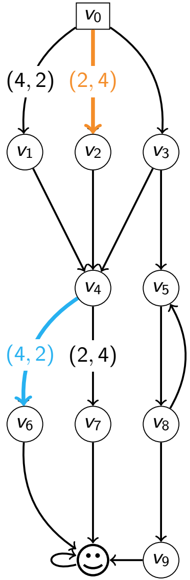
Player ◯ can **ensure** a cost profile $\mathbf{c} = (c_1, \ldots, c_d)$ from v if **there exists** a strategy $\sigma_{\bigcirc}$ such that **for all strategies** $\sigma_{\square}$ of Player □:

$$\text{Cost}(\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_v) \leq_C \mathbf{c} = (c_1, \ldots, c_d)$$

Ex:

- $(8, 8) \rightsquigarrow$ **Yes. (with memory!)**

  ▬ $\longmapsto$ ▬ $(8, 8)$



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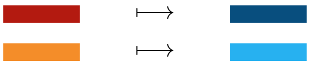

(8, 8)

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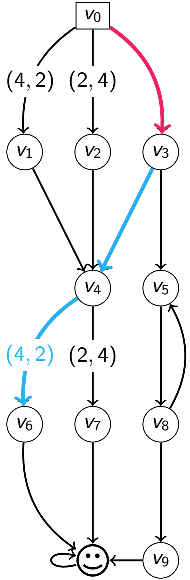

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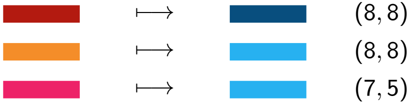

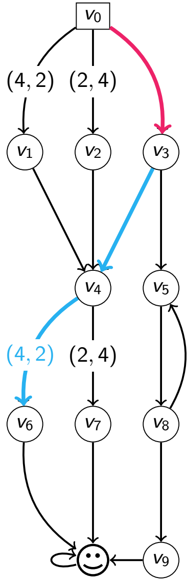

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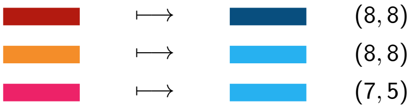


Player ◯ can adapt his strategy in function of the choice of Player □ $\rightsquigarrow$ finite-memory strategy!

$\mathsf{Ensure}(v) = \{\mathbf{c} \in \overline{\mathbb{N}}^d \mid \exists \sigma_{\bigcirc} \text{ st. } \forall \sigma_{\Box}, \mathsf{Cost}(\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_v) \leq_C \mathbf{c}\}$.

$\mathsf{minimal}(\mathsf{Ensure}(v)) = \mathsf{Pareto}(v) \rightsquigarrow$ **Pareto frontier** from v.

For $\mathbf{c} = (c_1, \ldots, c_d) \in \mathsf{Pareto}(v)$, a strategy $\sigma_{\bigcirc}$ is **c-Pareto-optimal** if $\sigma_{\bigcirc}$ ensures **c** from v.

Studied problems

Studied problems

1. Decide the **constrained existence problem**.
2. Compute the Pareto frontier and Pareto-optimal strategies.

Constrained existence problem (CEP)

Given a game, a vertex $v \in V$ and $\mathbf{c} \in \mathbb{N}^d$,
does there exist a strategy $\sigma_{\bigcirc}$ of Player $\bigcirc$ such that for all strategies of Player $\square$, we have:

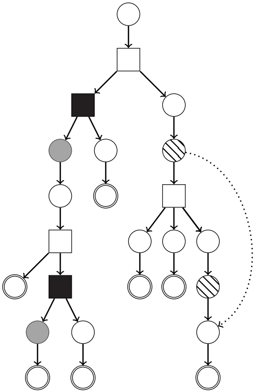
$$\mathrm{Cost}(\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_v) \leq_C \mathbf{c}$$

Constrained Existence Problem

Given $v \in V$ and $\mathbf{x} \in \mathbb{N}^d$, if there exists $\sigma_{\bigcirc}$ such that for all $\sigma_{\Box}$ we have: $\text{Cost}(\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_v) \leq_C \mathbf{x}$ then,

there exists $\sigma'_{\bigcirc}$ such that for all $\sigma_{\Box}$,

- $\text{Cost}(\langle \sigma'_{\bigcirc}, \sigma_{\Box} \rangle_v) \leq_C \mathbf{x}$;
- $|\langle \sigma'_{\bigcirc}, \sigma_{\Box} \rangle_v|_F \leq |V|$



$\rightsquigarrow$ simulation of the game by an alternating Turing machine during at most $|V|$ steps.

Since $\text{APTIME} = \text{PSPACE}$:

In two-player multi-weighted reachability games, the CE problem belongs to PSPACE.

In two-player multi-weighted reachability games, the CE problem is PSPACE-hard.

$\rightsquigarrow$ Reduction from the Quantified Subset-Sum problem.

Quantified Subset-Sum Problem

Given a set of natural numbers $N = \{a_1, \dots, a_n\}$ and a threshold $T \in \mathbb{N}$, we ask if the formula

$$\Psi = \exists x_1 \in \{0,1\} \, \forall x_2 \in \{0,1\} \, \exists x_3 \in \{0,1\} \dots \exists x_n \in \{0,1\}, \sum_{1 \leq i \leq n} x_i a_i = T$$

is true.

This problem is proved to be PSPACE-complete [Tra06, Lemma 4].

Computing the Pareto frontier

Computing Pareto(v)

$$\text{Ensure}^k(v) = \{\mathbf{c} \in \overline{\mathbb{N}}^d \mid \exists \sigma_{\bigcirc} \text{ st. } \forall \sigma_{\Box}, \text{Cost}(\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_v) \leq_C \mathbf{c} \wedge |\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_v|_F \leq k\}.$$

The algorithm computes, step by step, the sets $\mathrm{I}^k(v)$ for all $v \in V$.

For all $k \in \mathbb{N}$ and all $v \in V$, $\mathrm{I}^k(v) = \text{minimal}(\text{Ensure}^k(v))$

There exists $k^* \in \mathbb{N}$ such that for all $v \in V$ and for all $\ell \in \mathbb{N}$, $\mathrm{I}^{k^*}(v) = \mathrm{I}^{k^*+\ell}(v)$.

For all $v \in V$, $\mathrm{I}^{k^*}(v) = \text{Pareto}(v)$.

Theorem

The fixpoint algorithm runs in time polynomial in W and $|V|$ and is **exponential** in d,
where W is the maximal weight on an edge.

Computing Pareto(v)

for $v \in \mathsf{F}$ **do** $\mathsf{I}^0(v) = \{\mathbf{0}\}$
for $v \notin \mathsf{F}$ **do** $\mathsf{I}^0(v) = \{\infty\}$

repeat
for $v \in V$ **do**
if $v \in \mathsf{F}$ **then** $\mathsf{I}^{k+1}(v) = \{\mathbf{0}\}$

else if $v \in V_{\bigcirc}$ **then**

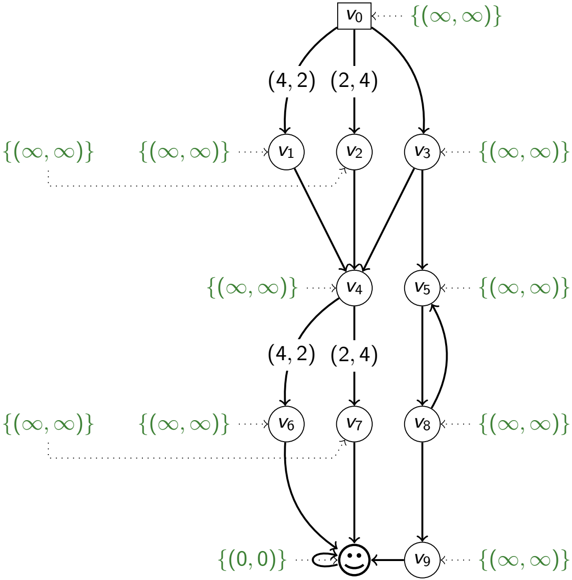
$$\mathsf{I}^{k+1}(v) = \text{minimal}\left(\bigcup_{v' \in \text{succ}(v)} \uparrow \mathsf{I}^k(v') + \mathbf{w}(v, v')\right)$$

else if $v \in V_{\Box}$ **then**

$$\mathsf{I}^{k+1}(v) = \text{minimal}\left(\bigcap_{v' \in \text{succ}(v)} \uparrow \mathsf{I}^k(v') + \mathbf{w}(v, v')\right)$$

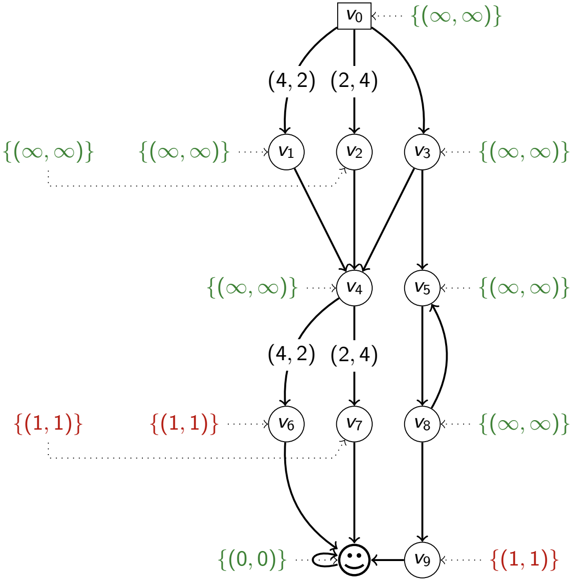
until $\mathsf{I}^{k+1}(v) = \mathsf{I}^k(v)$ for all $v \in V$

$I^0(\cdot)$

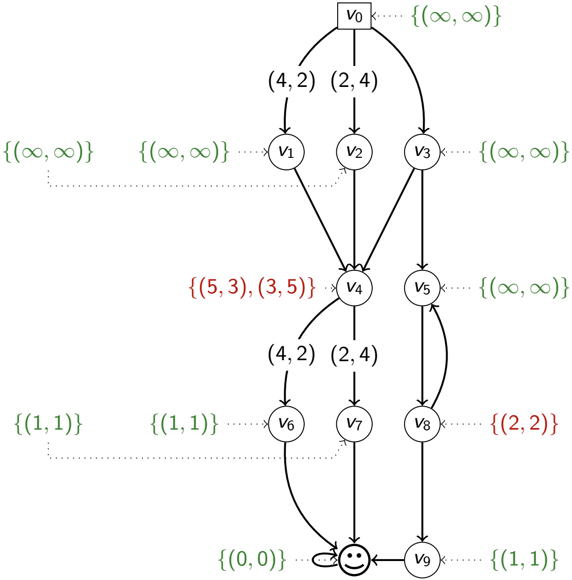


v_0
$\{(\infty,\infty)\}$
(4, 2)
(2, 4)
v_1
v_2
v_3
$\{(\infty,\infty)\}$
$\{(\infty,\infty)\}$
$\{(\infty,\infty)\}$
$\{(\infty,\infty)\}$
v_4
v_5
$\{(\infty,\infty)\}$
(4, 2)
(2, 4)
v_6
v_7
v_8
$\{(\infty,\infty)\}$
$\{(\infty,\infty)\}$
$\{(\infty,\infty)\}$
$\{(0,0)\}$
v_9
$\{(\infty,\infty)\}$

$I^1(\cdot)$

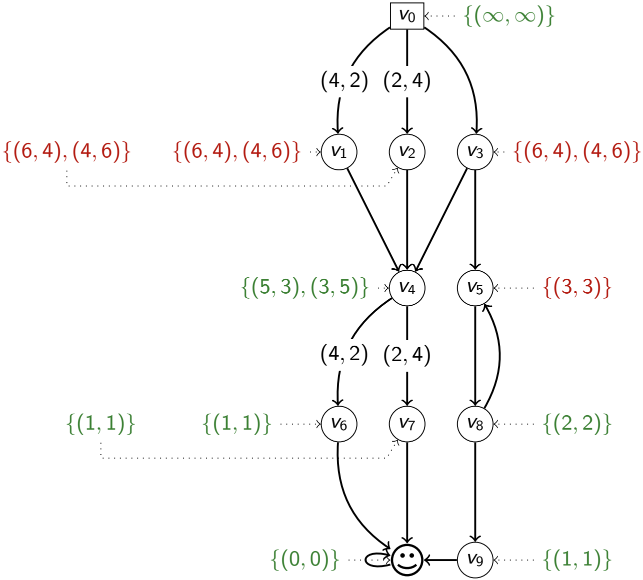


$I^2(\cdot)$



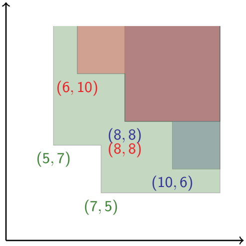
v_0
$\{(\infty, \infty)\}$
(4, 2)
(2, 4)
v_1
v_2
v_3
$\{(\infty, \infty)\}$
$\{(\infty, \infty)\}$
$\{(\infty, \infty)\}$
$\{(5, 3), (3, 5)\}$
v_4
v_5
$\{(\infty, \infty)\}$
(4, 2)
(2, 4)
$\{(1, 1)\}$
$\{(1, 1)\}$
v_6
v_7
v_8
$\{(2, 2)\}$
$\{(0, 0)\}$
v_9
$\{(1, 1)\}$

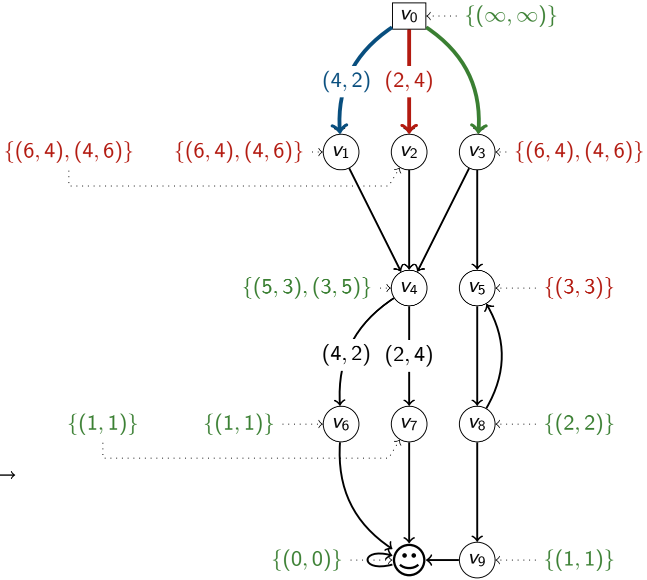
$I^3(\cdot)$



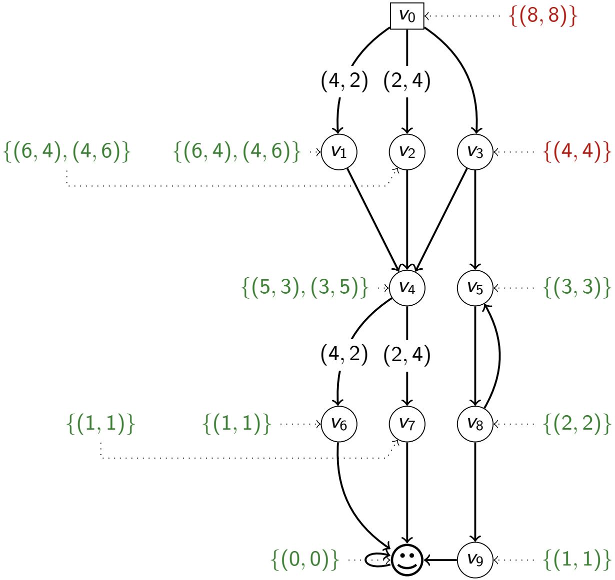
v_0
$\{(\infty,\infty)\}$
$(4,2)$
$(2,4)$
$\{(6,4),(4,6)\}$
$\{(6,4),(4,6)\}$
v_1
v_2
v_3
$\{(6,4),(4,6)\}$
$\{(5,3),(3,5)\}$
v_4
v_5
$\{(3,3)\}$
$(4,2)$
$(2,4)$
$\{(1,1)\}$
$\{(1,1)\}$
v_6
v_7
v_8
$\{(2,2)\}$
$\{(0,0)\}$
v_9
$\{(1,1)\}$

$I^3(\cdot)$


(6, 10)
(8, 8)
(8, 8)
(5, 7)
(10, 6)
(7, 5)

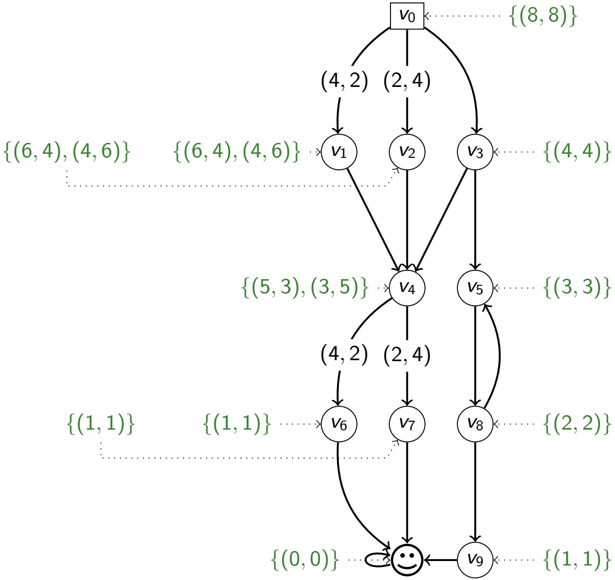

v0
{(∞, ∞)}
(4, 2)
(2, 4)
{(6, 4), (4, 6)}
{(6, 4), (4, 6)}
v1
v2
v3
{(6, 4), (4, 6)}
{(5, 3), (3, 5)}
v4
v5
{(3, 3)}
(4, 2)
(2, 4)
{(1, 1)}
{(1, 1)}
v6
v7
v8
{(2, 2)}
{(0, 0)}
v9
{(1, 1)}

$I^4(\cdot)$



v0
{(8, 8)}
(4, 2)
(2, 4)
{(6, 4), (4, 6)}
{(6, 4), (4, 6)}
v1
v2
v3
{(4, 4)}
{(5, 3), (3, 5)}
v4
v5
{(3, 3)}
(4, 2)
(2, 4)
{(1, 1)}
{(1, 1)}
v6
v7
v8
{(2, 2)}
{(0, 0)}
v9
{(1, 1)}

$\mathsf{I}^5(\cdot)$



Pareto-optimal strategies

for $v \in \mathrm{F}$ **do** $\mathrm{I}^0(v) = \{\mathbf{0}\}$
for $v \notin \mathrm{F}$ **do** $\mathrm{I}^0(v) = \{\infty\}$

repeat
  for $v \in V$ **do**
    if $v \in \mathrm{F}$ **then** $\mathrm{I}^{k+1}(v) = \{\mathbf{0}\}$

    else if $v \in V_{\bigcirc}$ **then**

$$\mathrm{I}^{k+1}(v) = \text{minimal}\left(\bigcup_{v' \in \text{succ}(v)} \uparrow \mathrm{I}^k(v') + \mathbf{w}(v, v')\right)$$

      for $\mathbf{x} \in \mathrm{I}^{k+1}(v)$ **do**
        if $\mathbf{x} \in \mathrm{I}^k(v)$ **then** $f_v^{k+1}(\mathbf{x}) = f_v^k(\mathbf{x})$
        else
          $f_v^{k+1}(\mathbf{x}) = (v', \mathbf{x}')$ where v' and $\mathbf{x}'$ are such that $v' \in \text{succ}(v)$, $\mathbf{x} = \mathbf{x}' + \mathbf{w}(v, v')$ and $\mathbf{x}' \in \mathrm{I}^k(v')$

    else if $v \in V_{\Box}$ **then**

$$\mathrm{I}^{k+1}(v) = \text{minimal}\left(\bigcap_{v' \in \text{succ}(v)} \uparrow \mathrm{I}^k(v') + \mathbf{w}(v, v')\right)$$

until $\mathrm{I}^{k+1}(v) = \mathrm{I}^k(v)$ for all $v \in V$

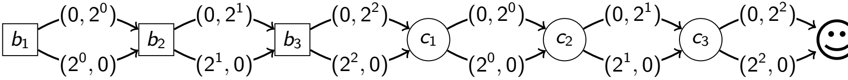
Computing Pareto-optimal strategies

Given $u \in V$ and $\mathbf{c} \in \mathrm{I}^*(u)\setminus\{\infty\}$, we define a strategy $\sigma^*_{\bigcirc}$ from u such that for all $hv \in \mathrm{Hist}_{\bigcirc}(u)$, let $\mathcal{C}(hv) = \{\mathbf{x}' \in \mathrm{I}^*(v) \mid \mathbf{x}' \leq_{\mathrm{C}} \mathbf{c} - \mathrm{Cost}(hv) \wedge \mathbf{x}' \leq_{\mathrm{L}} \mathbf{c} - \mathrm{Cost}(hv)\}$,

$$\sigma^*_{\bigcirc}(hv) = \begin{cases} v' & \text{for some } v' \in \mathrm{succ}(v), \text{ if } \mathcal{C}(hv) = \emptyset \\ f^*_v(\mathbf{x})[1] & \text{where } \mathbf{x} = \min_{\leq_{\mathrm{L}}} \mathcal{C}(hv), \text{ if } \mathcal{C}(hv) \neq \emptyset \end{cases}.$$

$\sigma^*_{\bigcirc}$ is a $\mathbf{c}$-Pareto-optimal strategy from u.

Memory requirements



Does there exist a strategy $\sigma_{\bigcirc}$ that ensures $(2^3 - 1, 2^3 - 1)$?

Intuitively:

- Player $\square$ generates two numbers on 3 bits: x and $\overline{x}$. Ex: $\downarrow\uparrow\downarrow \rightsquigarrow (x, \overline{x}) = (101, 010)$.
- Player $\bigcirc$ has to generate two numbers on 3 bits: y and $\overline{y}$ such that
  1. $x + y \leq 2^3 - 1$
  2. $\overline{x} + \overline{y} \leq 2^3 - 1$

  Ex: $\uparrow\downarrow\uparrow \rightsquigarrow (y, \overline{y}) = (010, 101)$ and so $x + y = \overline{x} + \overline{y} = 2^3 - 1$.
- Since $\overline{x} = (2^3 - 1) - x$, y should be equal to $\overline{x}$ to satisfy inequalities (1) and (2).
- Player $\square$ may generate all numbers between 0 and $2^3 - 1 \rightsquigarrow$ Player $\bigcirc$ has to answer differently with respect to the generated numbers $\rightsquigarrow 2^3$ combinations to keep in memory.

$\rightsquigarrow$ This example may be generalized to n bits $\rightsquigarrow$ we need strategies with **exponential memory**.

Conclusion

Conclusion

	Componentwise order	Lexicographic order
minimal(Ensure(v))	in exponential time	in polynomial time
CEP	PSPACE-complete	in P

- **uniform approach** to compute minimal(Ensure(v)) both for the componentwise order and the lexicographic order $\rightsquigarrow$ fixpoint algorithm;
- (Pareto)-optimal strategies can be synthesized thanks to the fixpoint algorithm;
- Pareto-optimal strategies may **require memory** (in the componentwise order case).

- Rem: The CEP with the componentwise order is undecidable with negative weights and four dimensions [Ran23].

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