# A stroll with reachability games

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Based on joint works with
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- Computing the Pareto frontier
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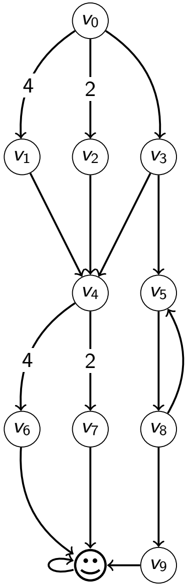
3 Multiplayer Reachability Games
- Nash equilibria
- Subgame perfect equilibria
- Conclusion

# Related Models

# One-player Reachability Games

# One-player Reachability Games




- A weighted graph $G = (V, E, w)$;
- One player: Player $\bigcirc$.

**Quantitative reachability objective**

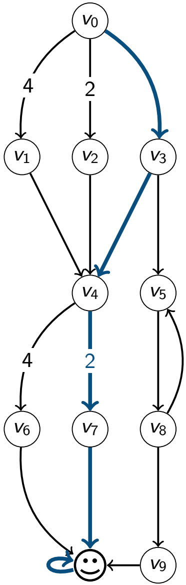
Given a target set F $\subseteq V$, for all **plays** (infinite paths in $G$) $\rho = \rho_0\rho_1 \ldots$:

$$\text{Cost}(\rho) = \begin{cases} \sum_{n=0}^{k-1} w(\rho_n, \rho_{n+1}) & \text{if } k \text{ is the least index st.} \rho_k \in T \\ +\infty & \text{otherwise} \end{cases}$$

Ex:

- $\text{Cost}(v_0 v_2 v_4 v_7 (☺)^\omega) = 6$;
- $\text{Cost}(v_0 v_3 (v_5 v_8)^\omega) = +\infty$

# Constrained existence




**Strategy:** $\sigma_{\bigcirc} : V^* V_{\bigcirc} \longrightarrow V$
**Outcome**:

- $\langle \sigma_{\bigcirc} \rangle_{v_0} \rightsquigarrow v_0 v_3 v_4 v_7 (☺)^{\omega}$;
- $\text{Cost}(\langle \sigma_{\bigcirc} \rangle_{v_0}) = 5$.

**Constrained existence (CE) problem**

Given $v \in V$ and $k \in \mathbb{N}$, does there exist $\sigma_{\bigcirc}$, such that

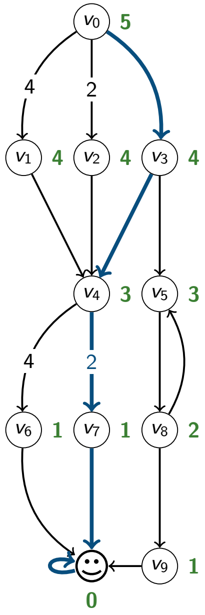
$$\text{Cost}(\langle \sigma_{\bigcirc} \rangle_v) \leq k?$$

Ex:

- with $k = 7$ and $v = v_0 \rightsquigarrow$ **YES**;
- with $k = 3$ and $v = v_0 \rightsquigarrow$ **NO**.

$\rightsquigarrow$ studying **shortest paths** in the game graph

# Optimality




**How to find shortest paths?**

- Dijkstra algorithm;
- Bellman-Ford algorithm;
- ...

**Main idea**

- $X(v) = 0$ if $v \in F$ and $= \infty$ otherwise
- Repeat: $X_{pre} = X$, for all $v \in V \setminus F$, $X(v) = \min_{v' \in \text{succ}(v)} \{X_{pre}(v') + w(v, v')\}$

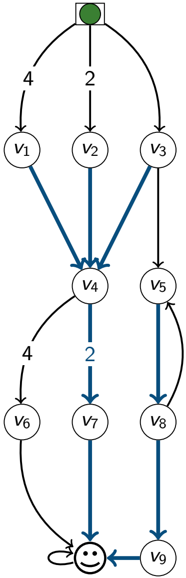
$\rightsquigarrow$ only computing some minimum.

In a one-player reachability game:

- the CE problem belongs to $\mathrm{P}$;
- computing the shortest path can be done in polynomial time.

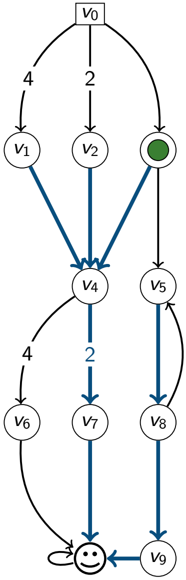
## Two-player Reachability Games

# Two-player Reachability Games



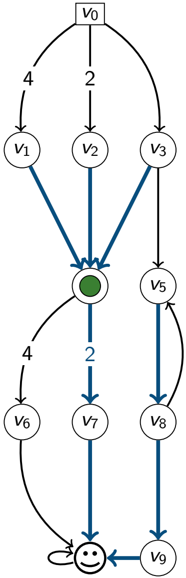

- A weighted graph $G = (V, E, w)$;
- Two players: Player ○ and Player □;
  - Player ○ wants to reach F ASAP;
  - Player □ wants to avoid that.

# Two-player Reachability Games



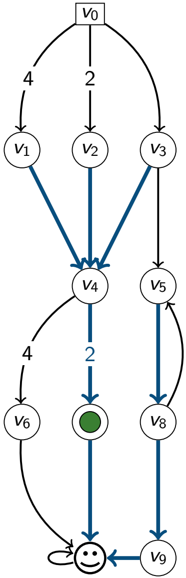

- A weighted graph $G = (V, E, w)$;
- Two players: Player ◯ and Player □;
  - Player ◯ wants to reach F ASAP;
  - Player □ wants to avoid that.

# Two-player Reachability Games



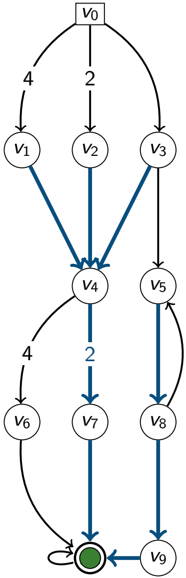

- A weighted graph $G = (V, E, w)$;
- Two players: Player ◯ and Player □;
  - Player ◯ wants to reach F ASAP;
  - Player □ wants to avoid that.

# Two-player Reachability Games



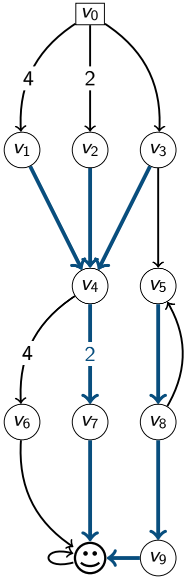

- A weighted graph $G = (V, E, w)$;
- Two players: Player $\bigcirc$ and Player $\square$;
  - Player $\bigcirc$ wants to reach F ASAP;
  - Player $\square$ wants to avoid that.

# Two-player Reachability Games




- A weighted graph $G = (V, E, w)$;
- Two players: Player $\bigcirc$ and Player $\square$;
  - Player $\bigcirc$ wants to reach F ASAP;
  - Player $\square$ wants to avoid that.

# Two-player Reachability Games




- A weighted graph $G = (V, E, w)$;
- Two players: Player ◯ and Player □;
  - Player ◯ wants to reach F ASAP;
  - Player □ wants to avoid that.

**Constrained existence (CE) problem**

Given $v \in V$ and $k \in \mathbb{N}$, does there exist $\sigma_{\bigcirc}$, such that for all $\sigma_{\Box}$,

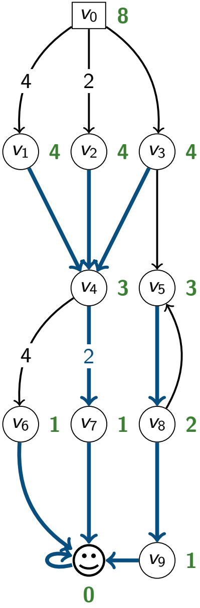
$$\text{Cost}(\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_v) \leq k$$

Ex:

- With $k = 7$, **NO!**
- With $k = 8$, **Yes.**

$\rightsquigarrow$ **value** $\text{Val}(v)$ of a vertex $v$.
$\rightsquigarrow$ **optimal strategies.**

# Optimality




**Main idea**

- If $v \in F$, $I^0(v) = 0$ and $= \infty$ otherwise.
- $k \rightsquigarrow k+1$. For all $v \in V$:
  - If $v \in V_{\bigcirc}$:
    $I^{k+1}(v) = \min_{v' \in \text{succ}(v)} (I^k(v) + w(v, v'))$.
  - If $v \in V_{\Box}$:
    $I^{k+1}(v) = \max_{v' \in \text{succ}(v)} (I^k(v) + w(v, v'))$.

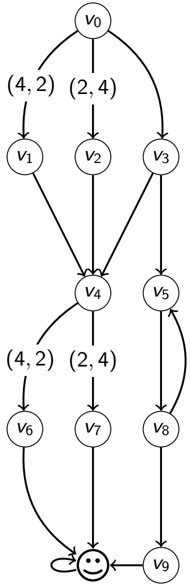
In a two-player reachability game:

- The CE problem belongs to P.
- Computing for all $v \in V$, Val($v$) can be done in polynomial time.
- There exist memoryless optimal strategies.

E.g.,[BGHM15]: To Reach or not to Reach? Efficient Algorithms for Total-Payoff Games, T. Brihaye at al.,

# One-player Multi-weighted Reachability Games

# One-Player Multi-Weighted Reachability Games




- A **d-weighted** graph $G = (V, E, (w_i)_{1 \leq i \leq d})$;
- A player: Player ◯;

### Quantitative reachability objective

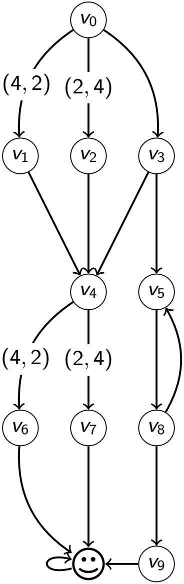
Given a target set F $\subseteq V$, for all plays $\rho = \rho_0\rho_1\ldots$ and all $1 \leq i \leq d$:

$$\text{Cost}_i(\rho) = \begin{cases} \sum_{n=0}^{k-1} w_i(\rho_n, \rho_{n+1}) & \text{if } k \text{ is the least index st.} \rho_k \in F \\ +\infty & \text{otherwise} \end{cases}$$

Rem: same target set F for all dimensions.

For all $\rho \in$ Plays, $\text{Cost}(\rho) = (\text{Cost}_i(\rho))_{1 \leq i \leq d}$.

# Constrained existence




**Componentwise order** $\leq_C$: for all $\mathbf{a}, \mathbf{b} \in \overline{\mathbb{N}}^d$,

$$a \leq_C b \Leftrightarrow a_i \leq b_i, \forall 1 \leq i \leq d$$

**Constrained existence (CE) problem**

Given $v \in V$ and $(k_1, \ldots, k_d) \in \mathbb{N}^d$, does there exist $\sigma_\bigcirc$, such that

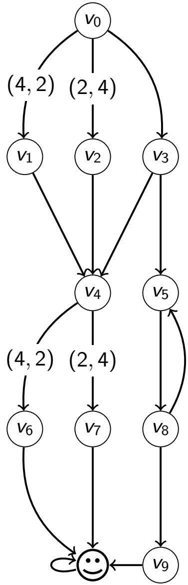
$$\text{Cost}(\langle \sigma_\bigcirc \rangle_v) \leq_C (k_1, \ldots, k_d)?$$

Ex:

- With $(k_1, k_2) = (8, 8)$ and and $v = v_0$: **YES**!
- With $(k_1, k_2) = (4, 4)$ and and $v = v_0$: **NO**!

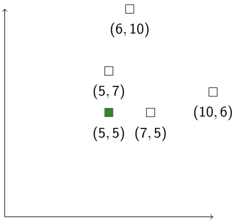
$$\text{Ensure}(v) = \{\mathbf{x} \in \overline{\mathbb{N}}^d \mid \exists \sigma_\bigcirc, \text{Cost}(\langle \sigma_\bigcirc \rangle_v) \leq_C \mathbf{x}\}.$$

# Optimality



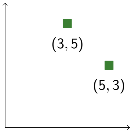

**What about optimality?**

- From $v_0$: $(10, 6)$, $(6, 10)$, $(7, 5)$, $(5, 7)$, $(5, 5)$, $(5 + 2n, 5 + 2n)$ for all $n \in \mathbb{N}$.




$\rightsquigarrow$ you would rather get $(5, 5)$.

- From $v_4$, do you prefer $(5, 3)$ or $(3, 5)$?



# Optimality

Let $X' \subseteq X$.

- $\text{minimal}(X') = \{x \in X' \mid (y \in X' \wedge y \leq_C x) \implies y = x\}$;
- $\uparrow X' = \{x \in X \mid \exists y \in X', y \leq_C x\}$;

**Optimality – Pareto frontier**

For $v \in V$, we want to compute the set:

$$\text{Pareto}(v) = \text{minimal}(\text{Ensure}(v))$$

# Optimality

**Main idea [PT02]**

- For all $v \in \mathrm{F}$, $\mathrm{I}^0(v) = 0^d$ and $= \infty^d$ otherwise;
- $k \rightsquigarrow k+1$: $v \in V$,

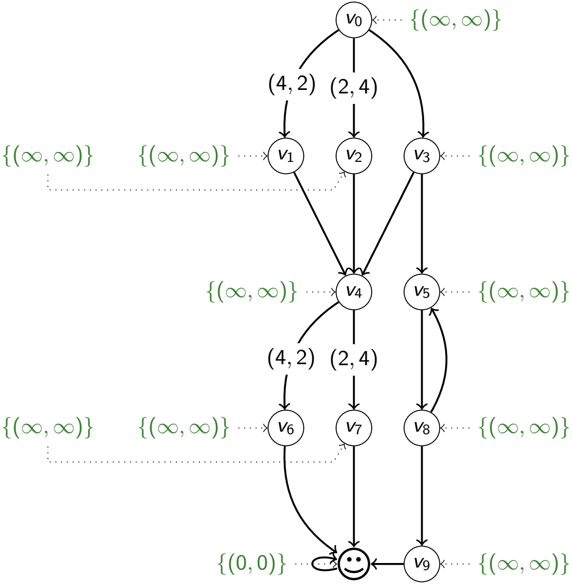
$$\mathrm{I}^{k+1}(v) = \begin{cases} 0^d & \textit{if } v \in \mathrm{F} \\ \text{minimal}\left( \bigcup_{v' \in \mathrm{succ}(v)} \mathrm{I}^k(v') + w(v, v') \right) & \text{otherwise} \end{cases}.$$

With

- $X + \mathbf{k} = \{\mathbf{x} + \mathbf{k} \mid \mathbf{x} \in X\}$
- for all $\mathbf{x}, \mathbf{y} \in \overline{\mathbb{N}}^d$, $\mathbf{z} = \mathbf{x} + \mathbf{y}$ is such that for all $1 \leq i \leq d$, $z_i = x_i + y_i$.

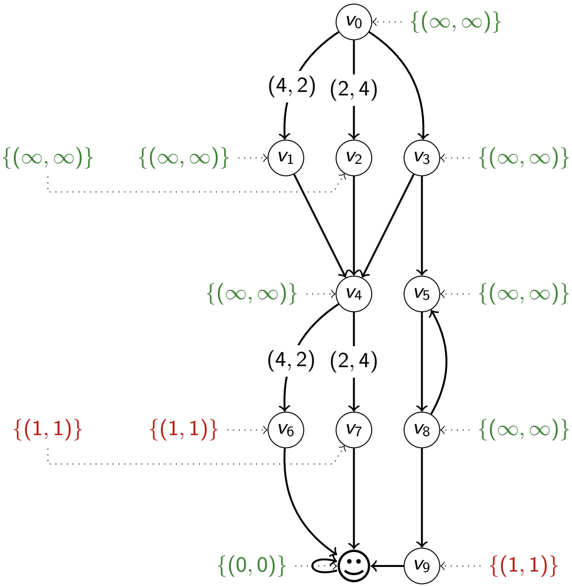
[PT02]: Algorithms for the Multi-constrained Routing Problem, Anuj Puri and Stavros Tripakis, SWAT 2002.

# Optimality



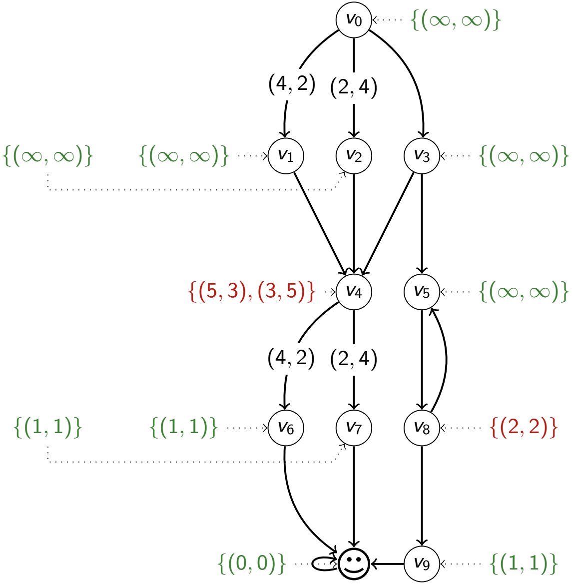
$v_0$
$\{(\infty,\infty)\}$
(4, 2)
(2, 4)
$v_1$
$v_2$
$v_3$
$\{(\infty,\infty)\}$
$\{(\infty,\infty)\}$
$\{(\infty,\infty)\}$
$v_4$
$v_5$
$\{(\infty,\infty)\}$
$\{(\infty,\infty)\}$
(4, 2)
(2, 4)
$v_6$
$v_7$
$v_8$
$\{(\infty,\infty)\}$
$\{(\infty,\infty)\}$
$\{(\infty,\infty)\}$
$v_9$
$\{(0,0)\}$
$\{(\infty,\infty)\}$

# Optimality

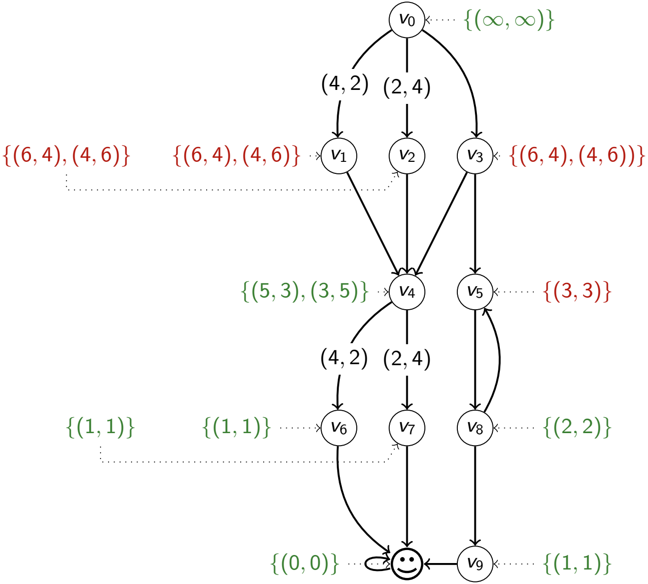


$v_0$
$\{(\infty, \infty)\}$
(4, 2)
(2, 4)
$\{(\infty, \infty)\}$
$\{(\infty, \infty)\}$
$v_1$
$v_2$
$v_3$
$\{(\infty, \infty)\}$
$\{(\infty, \infty)\}$
$v_4$
$v_5$
$\{(\infty, \infty)\}$
(4, 2)
(2, 4)
$\{(1, 1)\}$
$\{(1, 1)\}$
$v_6$
$v_7$
$v_8$
$\{(\infty, \infty)\}$
$\{(0, 0)\}$
$v_9$
$\{(1, 1)\}$

# Optimality


$\{(\infty,\infty)\}$
$v_0$
$(4,2)$
$(2,4)$
$\{(\infty,\infty)\}$
$\{(\infty,\infty)\}$
$v_1$
$v_2$
$v_3$
$\{(\infty,\infty)\}$
$\{(5,3),(3,5)\}$
$v_4$
$v_5$
$\{(\infty,\infty)\}$
$(4,2)$
$(2,4)$
$\{(1,1)\}$
$\{(1,1)\}$
$v_6$
$v_7$
$v_8$
$\{(2,2)\}$
$\{(0,0)\}$
$v_9$
$\{(1,1)\}$

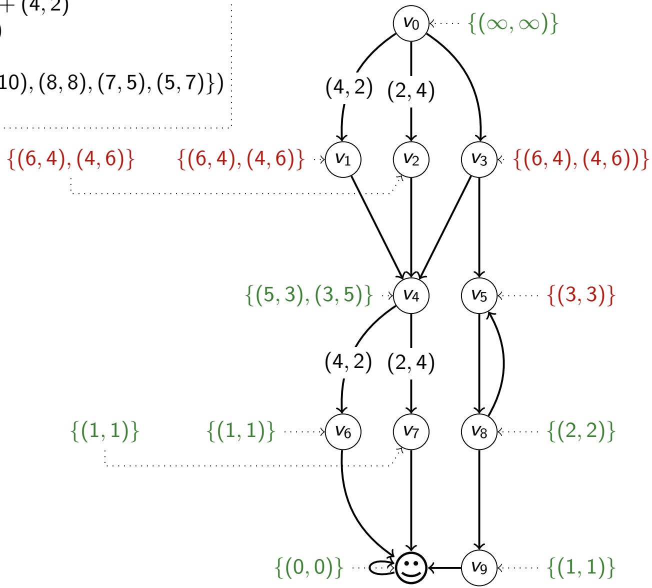
# Optimality



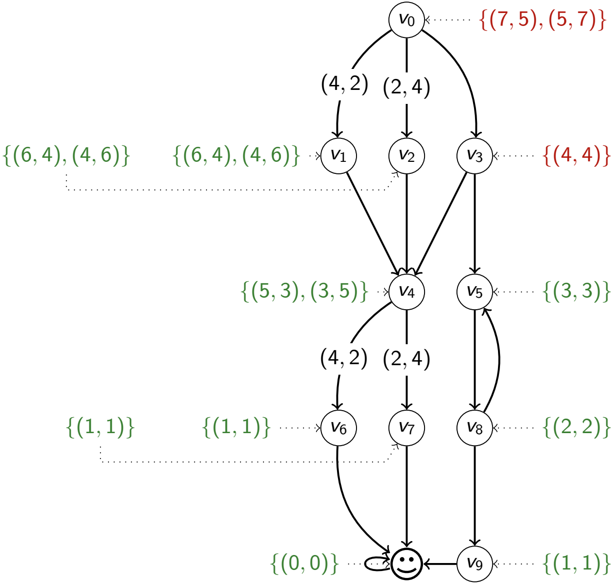
$v_0$
$\{(\infty,\infty)\}$
$(4,2)$
$(2,4)$
$v_1$
$v_2$
$v_3$
$\{(6,4),(4,6)\}$
$\{(6,4),(4,6)\}$
$\{(6,4),(4,6))\}$
$v_4$
$v_5$
$\{(5,3),(3,5)\}$
$\{(3,3)\}$
$(4,2)$
$(2,4)$
$v_6$
$v_7$
$v_8$
$\{(1,1)\}$
$\{(1,1)\}$
$\{(2,2)\}$
$v_9$
$\{(0,0)\}$
$\{(1,1)\}$

# Optimality

$$\begin{aligned}
&\text{minimal}(\{(6,4),(4,6)\} + (4,2)\\
&\cup\{(6,4),(4,6)\} + (2,4)\\
&\cup\{(6,4),(4,6) + (1,2))\\
&= \text{minimal}(\{(10,6),(6,10),(8,8),(7,5),(5,7)\})\\
&= \{(7,5),(5,7)\}
\end{aligned}$$

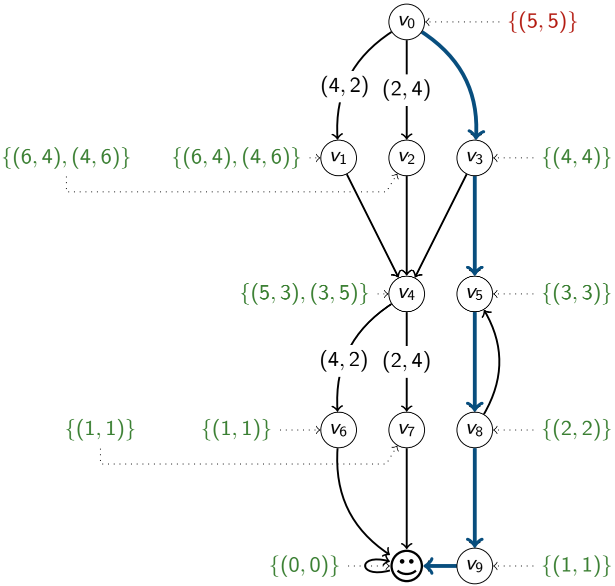


# Optimality



$v_0$
{(7, 5), (5, 7)}
(4, 2)
(2, 4)
{(6, 4), (4, 6)}
{(6, 4), (4, 6)}
$v_1$
$v_2$
$v_3$
{(4, 4)}
{(5, 3), (3, 5)}
$v_4$
$v_5$
{(3, 3)}
(4, 2)
(2, 4)
{(1, 1)}
{(1, 1)}
$v_6$
$v_7$
$v_8$
{(2, 2)}
{(0, 0)}
$v_9$
{(1, 1)}

# Optimality



{(5, 5)}
$v_0$
(4, 2)
(2, 4)
{(6, 4), (4, 6)}
{(6, 4), (4, 6)}
$v_1$
$v_2$
$v_3$
{(4, 4)}
{(5, 3), (3, 5)}
$v_4$
$v_5$
{(3, 3)}
(4, 2)
(2, 4)
{(1, 1)}
{(1, 1)}
$v_6$
$v_7$
$v_8$
{(2, 2)}
{(0, 0)}
$v_9$
{(1, 1)}

# Results

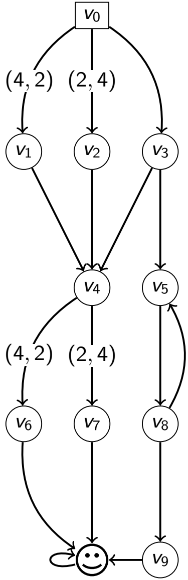
[PT02] In one-player multi-weighted reachability games:

- The CE problem is **NP-complete** .
- The algorithm to compute Pareto frontiers for all $v \in V$ is
  - polynomial in $\mathrm{W} = \max\{w \in \mathbb{N} \mid \exists 1 \leq i \leq d, \exists e \in E, w_i(e) = w\}$;
  - exponential in $d$.

[PT02]: Algorithms for the Multi-constrained Routing Problem, Anuj Puri and Stavros Tripakis, SWAT 2002.

# Two-player Multi-weighted Reachability Games

# Two-player Multi-weighted Reachability Games




- A multi-weighted graph $G = (V, E, (w_i)_{1 \leq i \leq d})$;
- Two players: Player $\bigcirc$ and Player $\Box$.

**Constrained existence (CE) problem**

Given $v \in V$ and $\mathbf{x} \in \overline{\mathbb{N}}^d$, does there exist $\sigma_{\bigcirc}$ st. for all $\sigma_{\Box}$,

$$\mathrm{Cost}(\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_v) \leq_C \mathbf{x}?$$

Rem: It is **not** possible to ensure $(5, 5)$.

# Optimality

$$\text{Ensure}(v) = \{\mathbf{x} \in \overline{\mathbb{N}}^d \mid \exists \sigma_{\bigcirc} \text{ st. } \forall \sigma_{\Box}, \text{Cost}(\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_v) \leq_C \mathbf{x}\}$$

**Optimality – Pareto frontier**

For $v \in V$, we want to compute $\text{Pareto}(v) = \text{minimal}(\text{Ensure}(v))$.

Given $v \in V$ and $\mathbf{x} \in \text{Pareto}(v)$,
$\sigma_{\bigcirc}$ is **x-Pareto-optimal** from $v$, if for all $\sigma_{\Box}$, $\text{Cost}(\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_v) \leq_C \mathbf{x}$

**Memory requirement for Pareto-optimal strategies**

Given $v \in V$ and $\mathbf{x} \in \text{Pareto}(v)$, which amount of memory is required by a $\mathbf{x}$-Pareto-optimal strategy?

# Studied problems

[BG23] In two-player multi-weighted reachability games:

- The CE problem is PSPACE-complete. (Rem: NP-complete if restricted to memoryless strategies.)
- Computing the Pareto frontier for all $v \in V$ can be done in time polynomial in the size of the graph and $W$ and exponential in $d$.
- Pareto-optimal strategies sometimes require memory.

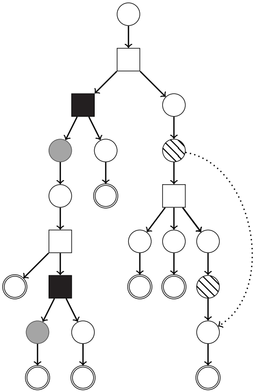
[BG23]: Multi-weighted Reachability Games, T. Brihaye and A. Goeminne, to appear in RP 2023.

# Constrained Existence Problem

Given $v \in V$ and $\mathbf{x} \in \mathbb{N}^d$, if there exists $\sigma_{\bigcirc}$ such that for all $\sigma_{\Box}$ we have: $\text{Cost}(\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_v) \leq_{\mathsf{C}} \mathbf{x}$ then,

there exists $\sigma'_{\bigcirc}$ such that for all $\sigma_{\Box}$,

- $\text{Cost}(\langle \sigma'_{\bigcirc}, \sigma_{\Box} \rangle_v) \leq_{\mathsf{C}} \mathbf{x}$;
- $|\langle \sigma'_{\bigcirc}, \sigma_{\Box} \rangle_v|_{\mathsf{F}} \leq |V|$



$\rightsquigarrow$ simulation of the game by an alternating Turing machine during at most $|V|$ steps.

Since $\text{APTIME} = \text{PSPACE}$:

In two-player multi-weighted reachability games, the CE problem belongs to PSPACE.

In two-player multi-weighted reachability games, the CE problem is PSPACE-hard.

$\rightsquigarrow$ Reduction from the Quantified Subset-Sum problem.

**Quantified Subset-Sum Problem**

Given a set of natural numbers $N = \{a_1, \dots, a_n\}$ and a threshold $T \in \mathbb{N}$, we ask if the formula

$$\Psi = \exists x_1 \in \{0,1\} \forall x_2 \in \{0,1\} \exists x_3 \in \{0,1\} \dots \exists x_n \in \{0,1\}, \sum_{1 \le i \le n} x_i a_i = T$$

is true.

This problem is proved to be PSPACE-complete [Tra06, Lemma 4].

## Computing the Pareto frontier

**Pareto frontier** from $v \rightsquigarrow \text{minimal}(\text{Ensure}(v)) = \text{Pareto}(v)$.

$$\text{Ensure}^k(v) = \{\mathbf{c} \in \overline{\mathbb{N}}^d \mid \exists \sigma_{\bigcirc} \text{ st. } \forall \sigma_{\Box}, \text{Cost}(\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_v) \leq_C \mathbf{c} \wedge |\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_v|_F \leq k\}.$$

The algorithm computes, step by step, the sets $\text{I}^k(v)$ for all $v \in V$.

For all $k \in \mathbb{N}$ and all $v \in V$, $\text{I}^k(v) = \text{minimal}(\text{Ensure}^k(v))$

There exists $k^* \in \mathbb{N}$ such that for all $v \in V$ and for all $\ell \in \mathbb{N}$, $\text{I}^{k^*}(v) = \text{I}^{k^*+\ell}(v)$.

For all $v \in V$, $\text{I}^{k^*}(v) = \text{Pareto}(v)$.

**Theorem**

The fixpoint algorithm runs in time polynomial in $W$ and $|V|$ and is **exponential** in $d$.

# Computing Pareto($v$)

**for** $v \in \mathsf{F}$ **do** $\mathsf{I}^0(v) = \{\mathbf{0}\}$
**for** $v \notin \mathsf{F}$ **do** $\mathsf{I}^0(v) = \{\infty\}$

**repeat**
  **for** $v \in V$ **do**
    **if** $v \in \mathsf{F}$ **then** $\mathsf{I}^{k+1}(v) = \{\mathbf{0}\}$

    **else if** $v \in V_{\bigcirc}$ **then**

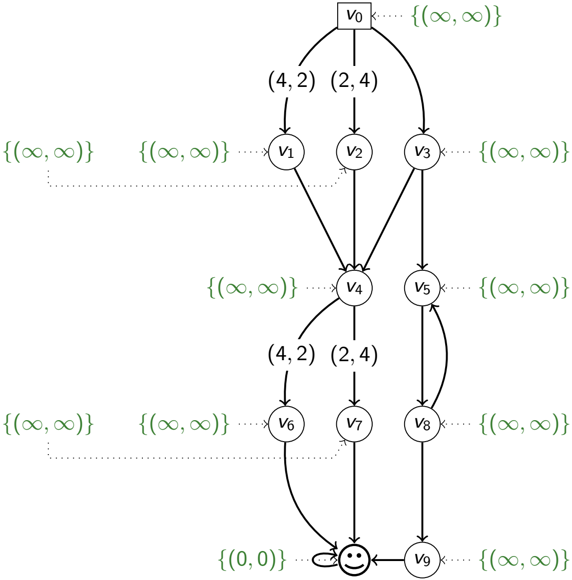
$$\mathsf{I}^{k+1}(v) = \text{minimal}\left(\bigcup_{v' \in \text{succ}(v)} \uparrow \mathsf{I}^k(v') + \mathbf{w}(v, v')\right)$$

    **else if** $v \in V_{\Box}$ **then**

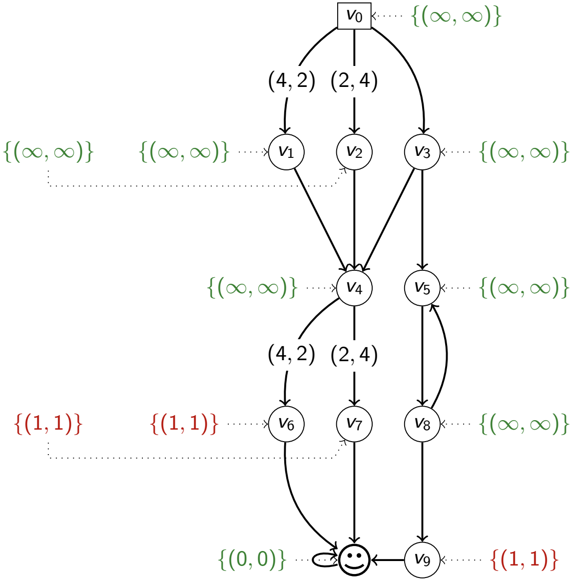
$$\mathsf{I}^{k+1}(v) = \text{minimal}\left(\bigcap_{v' \in \text{succ}(v)} \uparrow \mathsf{I}^k(v') + \mathbf{w}(v, v')\right)$$

**until** $\mathsf{I}^{k+1}(v) = \mathsf{I}^k(v)$ for all $v \in V$

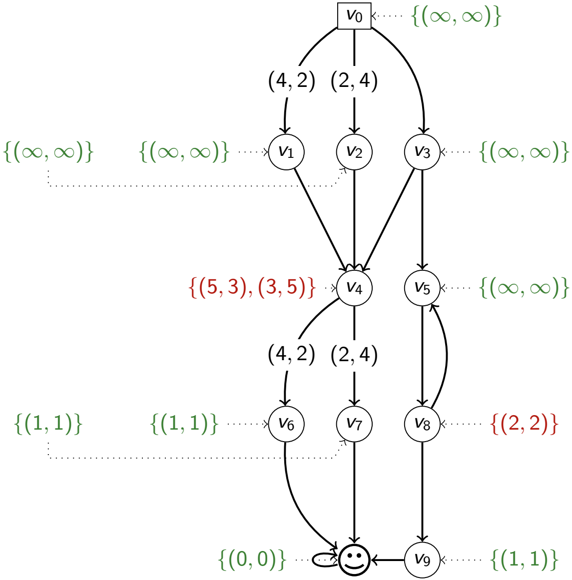
# $\mathsf{I}^0(\cdot)$



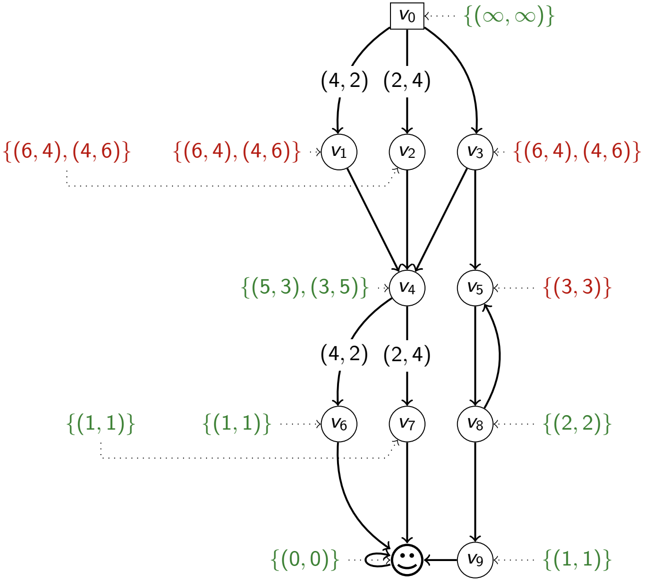
$\mathsf{I}^1(\cdot)$



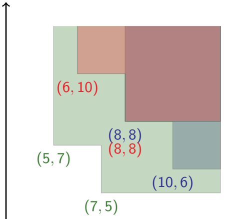
# $\mathsf{I}^2(\cdot)$

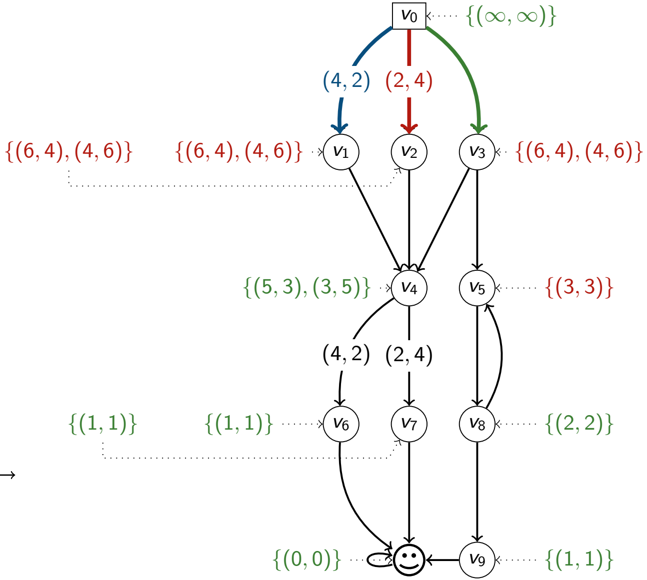


# $\mathsf{I}^3(\cdot)$

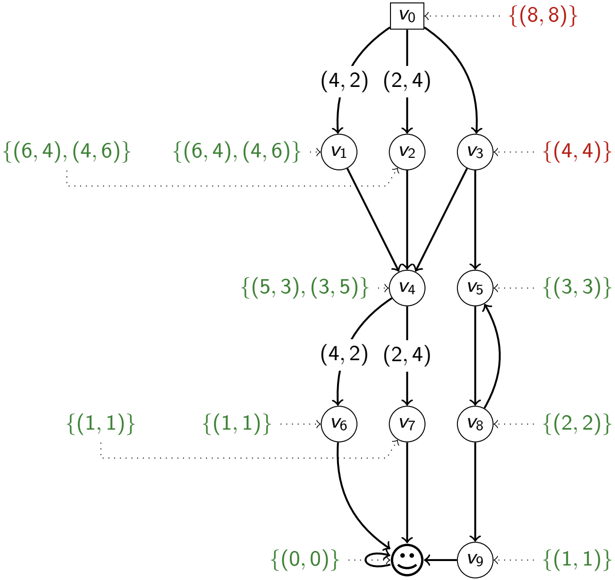


# $\mathsf{I}^3(\cdot)$

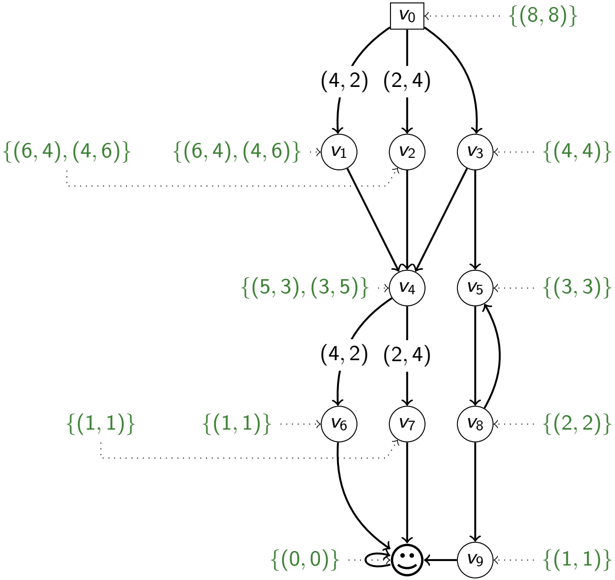


$I^4(\cdot)$


$v_0$
{(8, 8)}
(4, 2)
(2, 4)
{(6, 4), (4, 6)}
{(6, 4), (4, 6)}
$v_1$
$v_2$
$v_3$
{(4, 4)}
{(5, 3), (3, 5)}
$v_4$
$v_5$
{(3, 3)}
(4, 2)
(2, 4)
{(1, 1)}
{(1, 1)}
$v_6$
$v_7$
$v_8$
{(2, 2)}
{(0, 0)}
$v_9$
{(1, 1)}

$I^5(\cdot)$



v0
{(8, 8)}
(4, 2)
(2, 4)
{(6, 4), (4, 6)}
{(6, 4), (4, 6)}
v1
v2
v3
{(4, 4)}
{(5, 3), (3, 5)}
v4
v5
{(3, 3)}
(4, 2)
(2, 4)
{(1, 1)}
{(1, 1)}
v6
v7
v8
{(2, 2)}
{(0, 0)}
v9
{(1, 1)}

# Pareto-optimal strategies

**for** $v \in \mathsf{F}$ **do** $\mathsf{I}^0(v) = \{\mathbf{0}\}$
**for** $v \notin \mathsf{F}$ **do** $\mathsf{I}^0(v) = \{\infty\}$

**repeat**
  **for** $v \in V$ **do**
    **if** $v \in \mathsf{F}$ **then** $\mathsf{I}^{k+1}(v) = \{\mathbf{0}\}$

    **else if** $v \in V_{\bigcirc}$ **then**

$$\mathsf{I}^{k+1}(v) = \text{minimal}\left(\bigcup_{v' \in \mathsf{succ}(v)} \uparrow \mathsf{I}^k(v') + \mathbf{w}(v, v')\right)$$

      **for** $\mathbf{x} \in \mathsf{I}^{k+1}(v)$ **do**
        **if** $\mathbf{x} \in \mathsf{I}^k(v)$ **then** $f_v^{k+1}(\mathbf{x}) = f_v^k(\mathbf{x})$
        **else**
          $f_v^{k+1}(\mathbf{x}) = (v', \mathbf{x}')$ where $v'$ and $\mathbf{x}'$ are such that $v' \in \mathsf{succ}(v)$, $\mathbf{x} = \mathbf{x}' + \mathbf{w}(v, v')$ and $\mathbf{x}' \in \mathsf{I}^k(v')$

    **else if** $v \in V_{\square}$ **then**

$$\mathsf{I}^{k+1}(v) = \text{minimal}\left(\bigcap_{v' \in \mathsf{succ}(v)} \uparrow \mathsf{I}^k(v') + \mathbf{w}(v, v')\right)$$

**until** $\mathsf{I}^{k+1}(v) = \mathsf{I}^k(v)$ for all $v \in V$

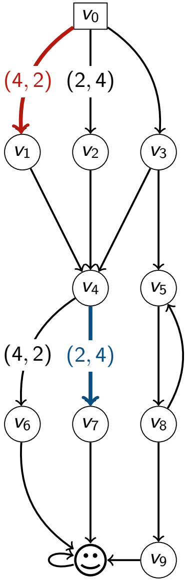
# Computing Pareto-optimal strategies

Given $u \in V$ and $\mathbf{c} \in \mathsf{I}^*(u) \setminus \{\infty\}$, we define a strategy $\sigma^*_{\bigcirc}$ from $u$ such that for all $hv \in \text{Hist}_{\bigcirc}(u)$, let $\mathcal{C}(hv) = \{\mathbf{x}' \in \mathsf{I}^*(v) \mid \mathbf{x}' \leq_{\mathsf{C}} \mathbf{c} - \text{Cost}(hv) \wedge \mathbf{x}' \leq_{\mathsf{L}} \mathbf{c} - \text{Cost}(hv)\}$,

$$\sigma^*_{\bigcirc}(hv) = \begin{cases} v' & \text{for some } v' \in \text{succ}(v), \text{ if } \mathcal{C}(hv) = \emptyset \\ f^*_v(\mathbf{x})[1] & \text{where } \mathbf{x} = \min_{\leq_{\mathsf{L}}} \mathcal{C}(hv), \text{ if } \mathcal{C}(hv) \neq \emptyset \end{cases}.$$

$\sigma^*_{\bigcirc}$ is a $\mathbf{c}$-Pareto-optimal strategy from $u$.

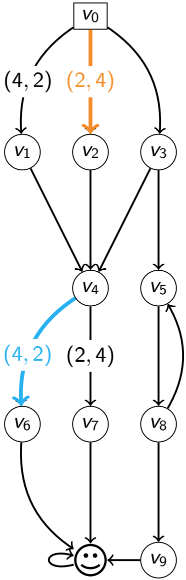
## Memory Requirements




- Is it possible to ensure $(8,8)$ from $v_0$? $\rightsquigarrow$ **Yes.** **(with memory!)**

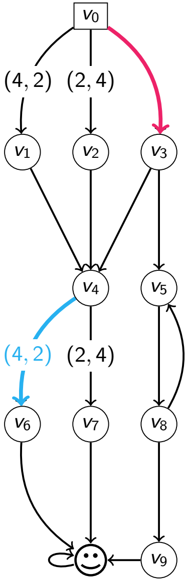
$\longmapsto$ $(8,8)$

Player ◯ can adapt his strategy in function of the choice of Player □ $\rightsquigarrow$ finite-memory strategy!

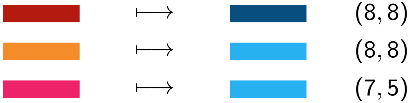



- Is it possible to ensure $(8, 8)$ from $v_0$? $\rightsquigarrow$ **Yes. (with memory!)**

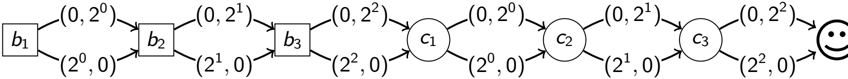

Player ◯ can adapt his strategy in function of the choice of Player □ $\rightsquigarrow$ finite-memory strategy!




- Is it possible to ensure $(8, 8)$ from $v_0$? $\rightsquigarrow$ **Yes. (with memory!)**




Player ◯ can adapt his strategy in function of the choice of Player □ $\rightsquigarrow$ finite-memory strategy!




Does there exist a strategy $\sigma_{\bigcirc}$ that ensures $(2^3 - 1, 2^3 - 1)$?

Intuitively:

- Player $\square$ generates two numbers on 3 bits: $x$ and $\overline{x}$. Ex: $\downarrow\uparrow\downarrow \rightsquigarrow (x, \overline{x}) = (101, 010)$.
- Player $\bigcirc$ has to generate two numbers on 3 bits: $y$ and $\overline{y}$ such that
  1. $x + y \leq 2^3 - 1$
  2. $\overline{x} + \overline{y} \leq 2^3 - 1$

  Ex: $\uparrow\downarrow\uparrow \rightsquigarrow (y, \overline{y}) = (010, 101)$ and so $x + y = \overline{x} + \overline{y} = 2^3 - 1$.
- Since $\overline{x} = (2^3 - 1) - x$, $y$ should be equal to $\overline{x}$ to satisfy inequalities (1) and (2).
- Player $\square$ may generate all numbers between 0 and $2^3 - 1 \rightsquigarrow$ Player $\bigcirc$ has to answer differently with respect to the generated numbers $\rightsquigarrow 2^3$ combinations to keep in memory.

$\rightsquigarrow$ This example may be generalized to $n$ bits $\rightsquigarrow$ we need strategies with **exponential memory**.

# Conclusion

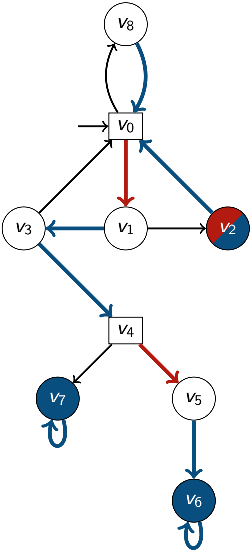
# Conclusion

| | Componentwise order | Lexicographic order |
|---|---|---|
| minimal(Ensure($v$)) | in exponential time | in polynomial time |
| **CEP** | PSPACE-complete | in P |

- **uniform approach** to compute minimal(Ensure($v$)) both for the componentwise order and the lexicographic order $\rightsquigarrow$ fixpoint algorithm;
- (Pareto)-optimal strategies can be synthesized thanks to the fixpoint algorithm;
- Pareto-optimal strategies may **require memory**.

# Multiplayer Reachability Games

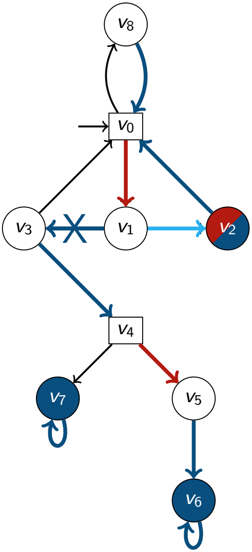
# Setting




- For all vertices $e$, $w(e) = 1$.
- An initial vertex: $v_0$;
- **Two** (or more) players;
  Ex: Player ○ and Player □.
- Objectives:
  - Player ○ wants to reach $F_{\bigcirc} = \{v_2, v_6, v_7\}$ (ASAP);
  - Player □ wants to **reach** $F_{\Box} = \{v_2\}$ (ASAP).
  - $\rightsquigarrow$ each player has his own target set.
- ~~optimal strategies~~ (optimality) $\rightsquigarrow$ equilibria (stability).

# Nash equilibria

# Definition



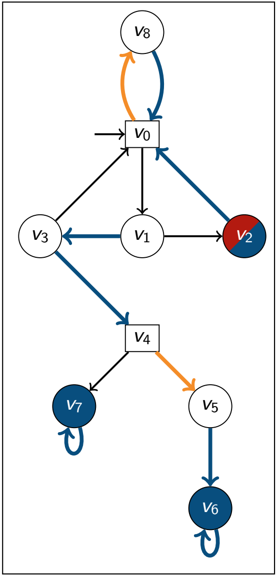

### Nash equilibrium

A **strategy profile** $(\sigma_{\bigcirc}, \sigma_{\Box})$ is a Nash equilibrium (NE) if **no** player has an **incentive to deviate unilaterally**.

- Counter-ex: $(\sigma_{\bigcirc}, \sigma_{\Box})$:
  - $(\sigma_{\bigcirc}, \sigma_{\Box}) \rightsquigarrow \langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_{v_0} = v_0 v_1 v_3 v_4 v_5 v_6^{\omega}$;
  - $(\mathrm{Cost}_{\bigcirc}(\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_{v_0}), \mathrm{Cost}_{\Box}(\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_{v_0})) = (5, +\infty)$.

$\rightsquigarrow$ not an NE.

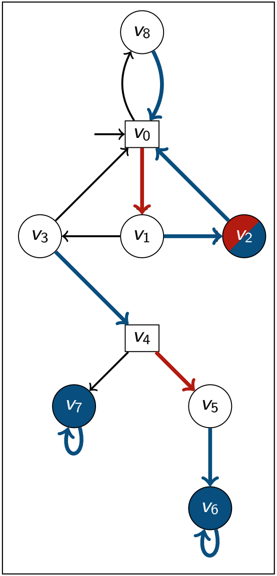
# Different NEs may coexist




- $\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0} = (v_0 v_8)^\omega$
- Cost : $(+\infty, +\infty)$
- **NO player** visits his target set ...

---

- $\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0} = (v_0 v_1 v_2)^\omega$
- Cost : $(2, 2)$
- **BOTH players** visit their target set !



What is (for us) a relevant **Nash equilibrium** ?

# Studied problems

1. **(Constrained existence problem)**
2. **(Social welfare decision problem)**
3. **(Pareto optimal decision problem)**

# Studied problems

1. **(Constrained existence problem)** Given $(k_1, \ldots, k_n) \in (\mathbb{N} \cup \{+\infty\})^n$, does there exist an NE $(\sigma_1, \ldots, \sigma_n)$ such that, for all $1 \leq i \leq n$:

$$\mathsf{Cost}_i(\langle \sigma_1, \ldots, \sigma_n \rangle_{v_0}) \leq k_i.$$

For NEs, in multiplayer quantitative reachability games, Problem 1 is **NP-complete**.[BBGT19]

[BBGT19]: On relevant equilibria in reachability games, T. Brihaye, V. Bruyère, A. Goeminne and N. Thomasset, RP'19.

# Key idea

**Outcome characterization of a Nash equilibrium**

Let $\rho$ be a play,

there exists an NE $(\sigma_1, \ldots, \sigma_n)$ such that $\langle \sigma_1, \ldots, \sigma_n \rangle_{v_0} = \rho$

if and only if

$\rho$ satisfies a "good" property.

# Key idea

**Outcome characterization of a Nash equilibrium**

Let $\rho$ be a play,

there exists an NE $(\sigma_1, \ldots, \sigma_n)$ such that $\langle \sigma_1, \ldots, \sigma_n \rangle_{v_0} = \rho$

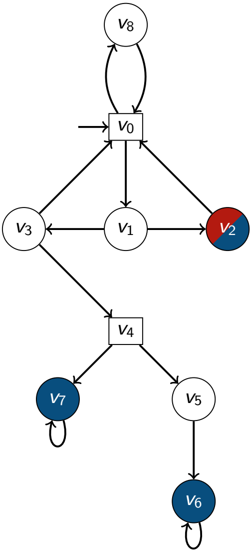
if and only if

$\rho$ satisfies a "good" property.

$\rightsquigarrow$ Does there exist a play $\rho$ such that:

- for each player $i$, $\text{Cost}_i(\rho) \leq k_i$;
- $\rho$ satisfies a "good" property?

# Outcome characterization of Nash equilibria

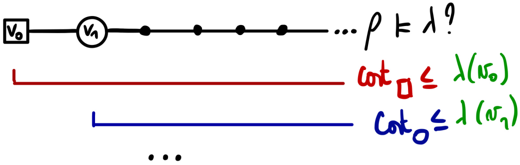



**What is this good property?**

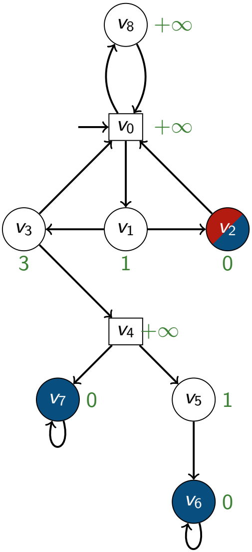
⇝ being $\lambda$-**consistent**.

**$\lambda$-consistent play**

- $\lambda : V \to \mathbb{N} \cup \{+\infty\}$: a labeling function;
- $\rho = \rho_0\rho_1 \ldots \models \lambda$ if and only if for all for all player $i$ and all $k \in \mathbb{N}$ such that $i \notin \text{Visit}(\rho_0 \ldots \rho_k)$ and $\rho_k \in V_i$: $\text{Cost}_i(\rho_{\geq k}) \leq \lambda(\rho_k)$.



# Outcome characterization of Nash equilibria




- $\lambda : V \to \mathbb{N} \cup \{+\infty\}$;
- $v_0 v_1 v_3 v_4 v_5 v_6^{\omega} \nvDash \lambda$:
  - $\mathrm{Cost}_{\square}(v_0 v_1 v_3 v_4 v_5 v_6^{\omega}) = +\infty \leq +\infty$ ✓
  - $\mathrm{Cost}_{\bigcirc}(v_1 v_3 v_4 v_5 v_6^{\omega}) = 4 \not\leq 1$ X
- $(v_0 v_8)^{\omega} \vDash \lambda$: $\mathrm{Cost} = (+\infty, +\infty)$;

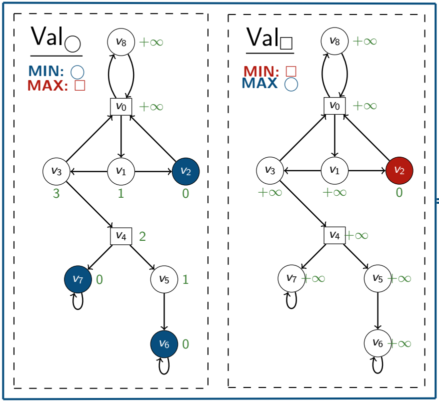
## How to find the good $\lambda$ ?

**Main idea:** $\lambda(v)$: the maximal number of steps within which the player who owns this vertex should reach his target set along $\rho$, starting from $v$.

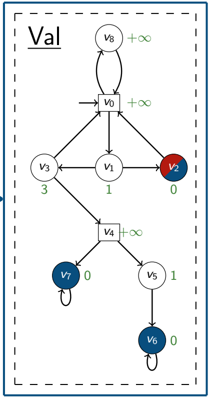
## NE outcome characterization [BBGT19]

A play $\rho$ is the outcome of an NE
if and only if
$\rho$ is Val-consistent.

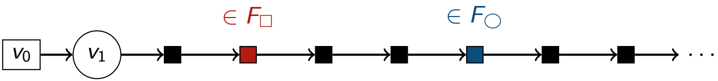
$$\mathrm{Val}(v) = \begin{cases} \mathrm{Val}_{\bigcirc}(v) & \text{if } v \in V_{\bigcirc} \\ \mathrm{Val}_{\Box}(v) & \text{if } v \in V_{\Box} \end{cases}.$$

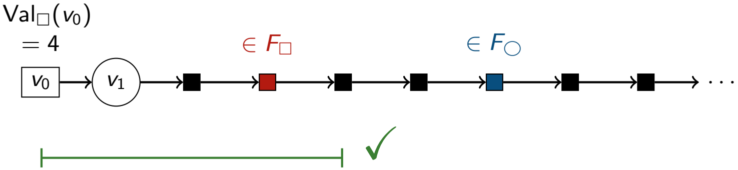



**2**
Two player
zero-sum
games

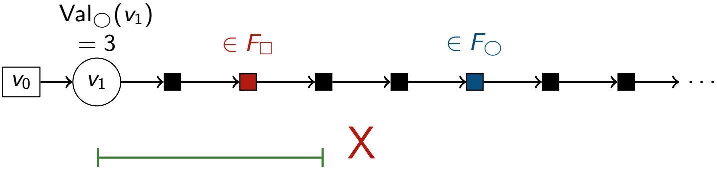



**1**
two player
(non zero-sum)
game

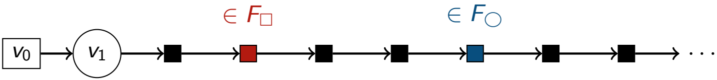

$\in F_\Box$
$\in F_\bigcirc$
$v_0$
$v_1$
$\cdots$



$\mathrm{Val}_{\Box}(v_0)$
$= 4$
$\in F_{\Box}$
$\in F_{\bigcirc}$
$v_0$
$v_1$
$\cdots$

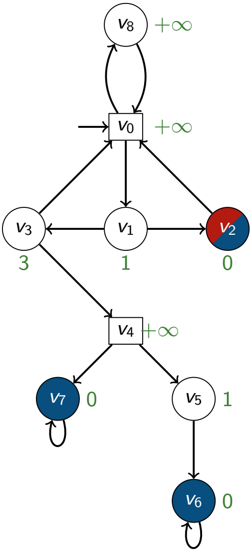


$\mathrm{Val}_{\bigcirc}(v_1)$
$= 3$
$\in F_{\Box}$
$\in F_{\bigcirc}$
$v_0$
$v_1$
$\cdots$
X




- ✓ ✓ ✓ ✓ ✓ . . . $\rightsquigarrow$ outcome of an NE;
- ✓ ✓ X $\rightsquigarrow$ ~~outcome of an NE~~.

# Outcome characterization of Nash equilibria



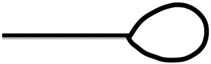

- $\not\exists\, \mathsf{Val} : V \to \mathbb{N} \cup \{+\infty\}$;
- $v_0 v_1 v_3 v_4 v_5 v_6^\omega \not\models \mathsf{Val}$:
  - $\mathsf{Cost}_\square(v_0 v_1 v_3 v_4 v_5 v_6^\omega) = +\infty \leq +\infty$ ✓
  - $\mathsf{Cost}_\bigcirc(v_1 v_3 v_4 v_5 v_6^\omega) = 4 \not\leq 1$ X
- $(v_0 v_8)^\omega \models \mathsf{Val}$: $\mathsf{Cost} = (+\infty, +\infty)$;

# Algorithm (For NE)

1. it guesses a lasso of polynomial length;
2. it verifies that the cost profile of this lasso satisfies the conditions given by the problem;
3. it verifies that the lasso is the outcome of an NE.

**NP-algorithm for Problem 1:**

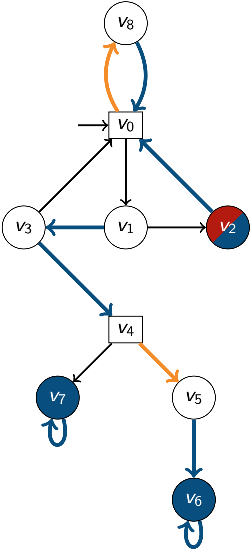
- **Step 1:** if there exists an NE which satisfies the constraints, there exists one which also satisfies the constraints and such that its outcome is a **lasso** ($h\ell^{\omega}$) with a



  **polynomial length** ($|h\ell|$).
- **Step 2:** can be done in **polynomial time.**
- **Step 3:** checking the Val-consistence along the lasso of polynomial length can be done in **polynomial time.**

# Subgame perfect equilibria

# Definition of subgame perfect equilibrium

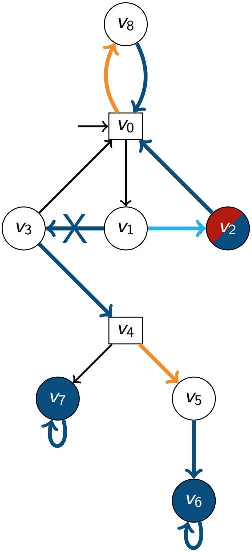


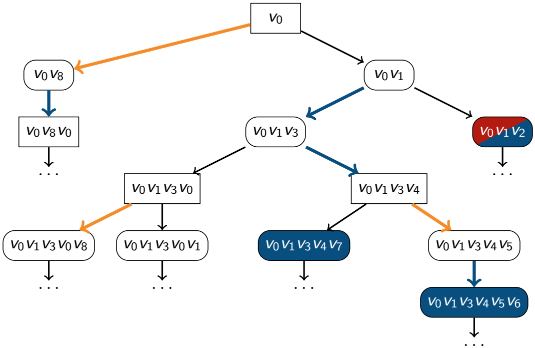

- refined solution concept: **subgame perfect equilibrium**.

### Subgame perfect equilibrium

A strategy profile $(\sigma_{\bigcirc}, \sigma_{\square})$ is a subgame perfect equilibrium (SPE) if it is an NE from each history.

# Definition of subgame perfect equilibrium




- $(\sigma_{\bigcirc}, \sigma_{\Box})$ is an **NE**;
- $(\sigma_{\bigcirc}, \sigma_{\Box})$ is **not an SPE**:
  there is a **profitable deviation** from $v_0 v_1$.

# (The same) Studied problems

1. **(The constrained existence problem)** Given $(k_1, \ldots, k_n) \in (\mathbb{N} \cup \{+\infty\})^n$, does there exist an ~~NE~~ **SPE** $(\sigma_1, \ldots, \sigma_n)$ such that, for all $1 \leq i \leq n$:
$$\text{Cost}_i(\langle \sigma_1, \ldots, \sigma_n \rangle_{v_0}) \leq k_i.$$
2. **(Social welfare decision problem)**
3. **(Pareto optimal decision problem)**

# (The same) Studied problems

1. **(The constrained existence problem)** Given $(k_1, \ldots, k_n) \in (\mathbb{N} \cup \{+\infty\})^n$, does there exist an ~~NE~~ **SPE** $(\sigma_1, \ldots, \sigma_n)$ such that, for all $1 \leq i \leq n$:

$$\text{Cost}_i(\langle \sigma_1, \ldots, \sigma_n \rangle_{v_0}) \leq k_i.$$

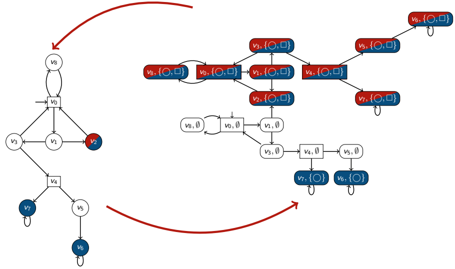
For ~~NEs~~ **SPEs**, in multiplayer quantitative reachability games, Problem 1 is ~~NP-complete~~ **PSPACE-complete**.[BBG+19]

[BBG+19]: The complexity of subgame perfect equilibria in quantitative reachability games, T. Brihaye, V. Bruyère, A. Goeminne, J.-F. Raskin, and M. van den Bogaard, CONCUR'19.

# (The same) Key idea

## SPE outcome characterization

A play $\rho$ is the outcome of an SPE
if and only if
$\rho$ is $\lambda^*$-consistent.




$\rightsquigarrow$ $\lambda^*$: the fixpoint of this algorithm:

## Computation of $\lambda^*$

$k \leftarrow 0$
**foreach** $(v, I) \in V^X$ (with $(v, I) \in V_i$ for some player $i$) **do**
  **if** $i \in I$ **then**
    $\lambda^0(v, I) = 0$
  **else**
    $\lambda^0(v, I) = +\infty$
  **end**
**end**
**repeat**
  $k \leftarrow k + 1$
  **foreach** $(v, I) \in V^X$ (with $(v, I) \in V_i$ for some player $i$) **do**
    **if** $i \in I$ **then**
      $\lambda^k(v, I) = 0$
    **else**
      $\lambda^{k+1}(v, I) = 1 + \min_{(v', I') \in \mathrm{Succ}(v, I)} \max\{\mathrm{Cost}_i(\rho) \mid \rho \in \Lambda^k(v', I')\}$
    **end**
  **end**
**until** $\lambda^k = \lambda^{k-1}$
return $\lambda^k$.

# Conclusion

# Conclusion

- characterization of the **complexity** of several decision problems related to the existence of **relevant equilibria**: in quantitative and qualitative Reachability games:

  Problem 1 : **the constrained existence problem** (CE);
  Problem 2 : the social welfare decision problem (SW);
  Problem 3 : the Pareto optimal decision problem (PO);

| Comp. | Qual. Reach. | | Quant. Reach. | |
|---|---|---|---|---|
| | NE | SPE | NE | SPE |
| CE | NP-c [CFGR16] | PSPACE-c [BBGR18] | NP-c | PSPACE-c [BBG+19] |
| SW | NP-c | PSPACE-c | NP-c | PSPACE-c |
| PO | NP-h/$\Sigma_2^P$ | PSPACE-c | NP-h/$\Sigma_2^P$ | PSPACE-c |

| Memory | Qual. Reach. | | Quant. Reach. | |
|---|---|---|---|---|
| | NE | SPE | NE | SPE |
| CE | Poly.[CFGR16] | Expo. [BBGR18] | Poly. | Expo. |
| SW | Poly. | Expo. | Poly. | Expo. |
| PO | Poly. | Expo. | Poly. | Expo. |

For more details: [BBGT19]: Thomas Brihaye, Véronique Bruyère, Aline Goeminne, Nathan Thomasset, On relevant equilibria in reachability games, RP 2019; or [BBGT21].

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