Constrained Existence Problem for Weak Subgame Perfect Equilibria with ω-regular Boolean Objectives

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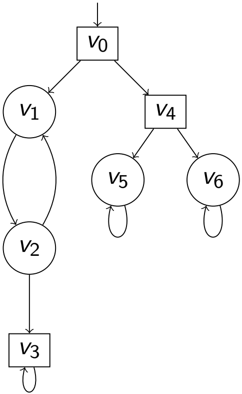
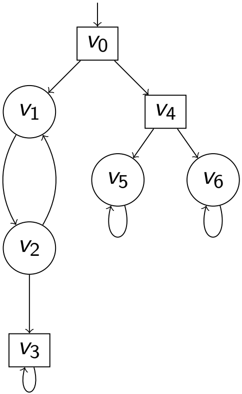
Theoretical background and studied problem

Boolean games



- $G = (V, E)$: a finite directed graph;
- Π: a finite set of players;
- $(V_i)_{i \in \Pi}$: a partition of V between the players;
- for each $i \in \Pi$, $\text{Gain}_i : V^\omega \to \{0, 1\}$: gain function;
- initialized game $(\mathcal{G}, v_0)$.

Plays and histories



- *play* ρ: infinite path in G from v_0;
  Ex : $v_0 v_1 v_2 v_3^\omega$
- *history* h: finite path in G from v_0;
  Ex: $v_0 v_1$

Objectives (1/2)

Given a play, how to define the gain of a player ?

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- Each player $i \in \Pi$ has an ω-regular objective charaterized by $\mathsf{Win}_i \subseteq V^\omega$.
- $\mathsf{Gain}_i(\rho) = 1$ if and only if $\rho \in \mathsf{Win}_i$.

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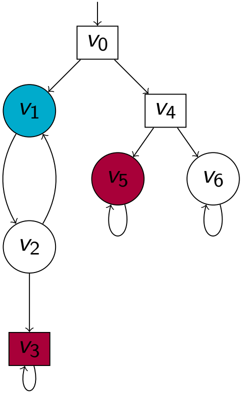
Classical ω-regular objectives: Büchi, co-Büchi, Explicit Muller, Muller, Parity, Streett and Rabin.

Rem:

- prefix-independent objectives;
- all players have the same type of objective (ex: each player has a Büchi objective).

Objectives (2/2)

Example

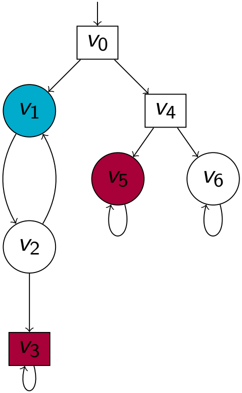


Game with Büchi objectives:

- $P_\bigcirc$: $\{v_1\}$;
- $P_\Box$: $\{v_3, v_5\}$;

Objectives (2/2)

Example

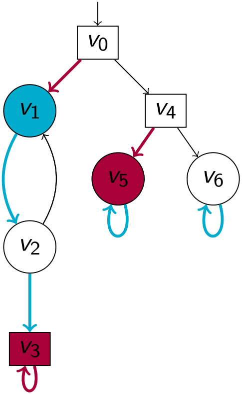


Game with Büchi objectives:

- $P_{\bigcirc}$: $\{v_1\}$;
- $P_{\Box}$: $\{v_3, v_5\}$;
- Play $\rho = v_0 v_1 v_2 v_3^{\omega}$:
  $\text{Gain}_{\bigcirc}(\rho) = 0$
  $\text{Gain}_{\Box}(\rho) = 1$.

Strategies (1/2)

A *strategy* can be associated with each player $i \in \Pi$: $\sigma_i : \mathrm{Hist}_i(v_0) \to V$.



- $\sigma_{\bigcirc}$: memoryless strategy of player $P_{\bigcirc}$;
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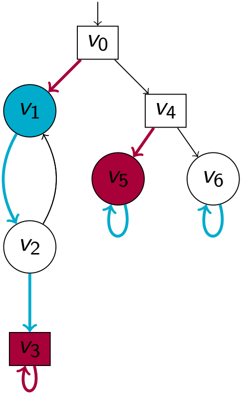
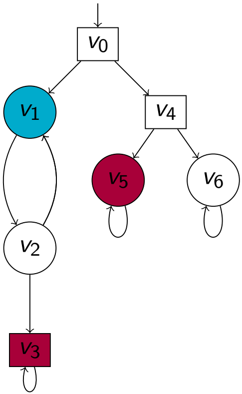


- $\sigma_{\bigcirc}$: memoryless strategy of player $P_{\bigcirc}$;
- $\sigma_{\Box}$: memoryless strategy of player $P_{\Box}$;
- $(\sigma_{\bigcirc}, \sigma_{\Box})$ is a *strategy profile*, denoted by $\overline{\sigma}$;
- *outcome*: $\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_{v_0} = v_0 v_1 v_2 v_3^{\omega}$.

Strategies (2/2)

Finitely deviating strategy



Finitely deviating strategy

Let σ_i and σ'_i be two strategies, σ'_i is *finitely deviating* from σ_i if σ'_i and σ_i **differ** only on a **finite number of histories**.

Strategies (2/2)

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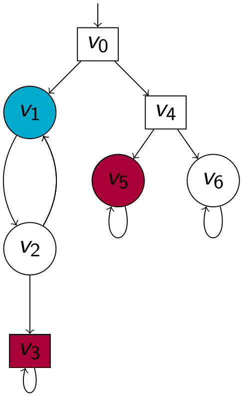

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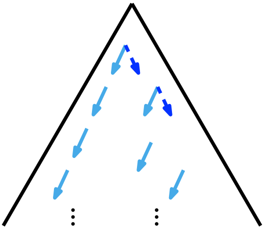

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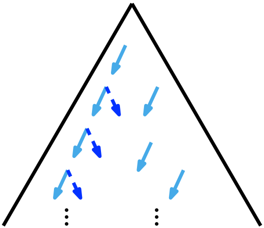
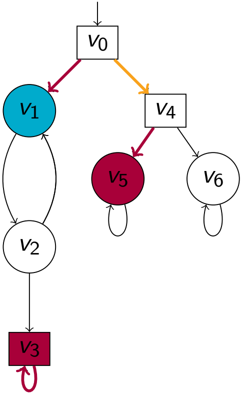


Not finitely deviating

Strategies (2/2)

Finitely deviating strategy

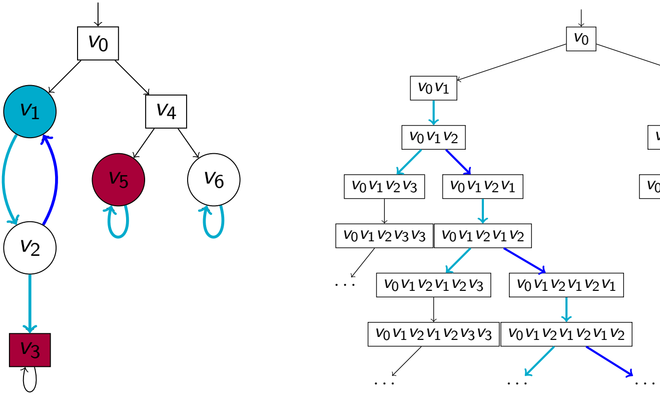
Example: $\sigma'_{\Box}$ differs from $\sigma_{\Box}$ only on the history v_0: $\sigma'_{\Box}(v_0) = v_4$ and $\sigma_{\Box}(v_0) = v_1$.



Strategies (2/2)

Finitely deviating strategy

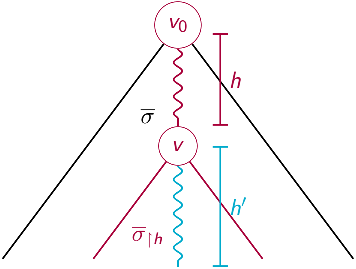
Counter-example: $\sigma'_{\bigcirc}$ differs from $\sigma_{\bigcirc}$ on an infinite number of histories: $v_0(v_1 v_2)^k$ $(k \in \mathbb{N}_0)$: $\sigma'_{\bigcirc}(v_0(v_1 v_2)^k) = v_1$ and $\sigma_{\bigcirc}(v_0(v_1 v_2)^k) = v_3$.



Subgame

Let $hv \in \text{Hist}(v_0)$ be a history, the game $(\mathcal{G}_{\restriction h}, v)$ called *subgame* of $(\mathcal{G}, v_0)$ is the same game played after hv:

- $\text{Gain}_{\restriction h}(\rho) = \text{Gain}(h\rho)$;
- if $\overline{\sigma} \to \overline{\sigma}_{\restriction h}$ such that $h' \in \text{Hist}(v)$: $\overline{\sigma}_{\restriction h}(h') = \overline{\sigma}(hh')$.



NE, SPE, weak NE and weak SPE (1/2)

Nash Equilibrium

Let $\overline{\sigma}$ be a strategy profile, $\overline{\sigma}$ is a *Nash equilibrium* (NE) in $(\mathcal{G}, v_0)$, if for all $i \in \Pi$ and σ'_i :

$$\text{Gain}_i(\langle \overline{\sigma} \rangle_{v_0}) \geq \text{Gain}_i(\langle \sigma'_i, \sigma_{-i} \rangle_{v_0}).$$

Rem: no profitable deviation

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(Weak) Nash Equilibrium

Let $\overline{\sigma}$ be a strategy profile, $\overline{\sigma}$ is a weak *Nash equilibrium* (weak NE) in $(\mathcal{G}, v_0)$, if for all $i \in \Pi$ and σ'_i finitely deviating from σ_i:

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subgame perfect equilibrium

Let $\overline{\sigma}$ be a strategy profile, $\overline{\sigma}$ is a *subgame perfect equilibrium* (SPE) in $(\mathcal{G}, v_0)$, if for all $hv \in \mathrm{Hist}(v_0)$: $\overline{\sigma}_{\restriction h}$ is a NE in $(\mathcal{G}_{\restriction h}, v)$.

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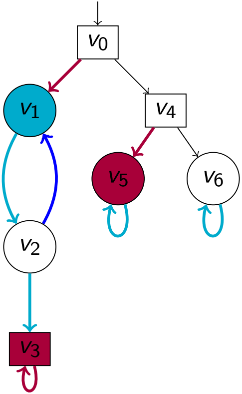
Notions of weak NE/SPE already introduced and studied in [BBMR15] and [BRPR17].

NE, SPE, weak NE and weak SPE (2/2)

Example



- $\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_{v_0} = v_0 v_1 v_2 v_3^{\omega}$: Gain $= (0, 1)$;

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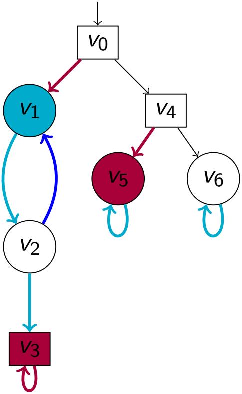

- $\langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_{v_0} = v_0 v_1 v_2 v_3^{\omega}$: Gain $= (0, 1)$;
- profitable deviation $\sigma'_{\bigcirc}$ for $P_{\bigcirc}$, $\langle \sigma'_{\bigcirc}, \sigma_{\Box} \rangle_{v_0} = v_0 (v_1 v_2)^{\omega}$, Gain $= (1, 0)$

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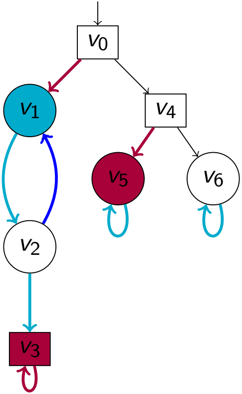

- $\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0} = v_0 v_1 v_2 v_3^{\omega}$: Gain $= (0, 1)$;
- profitable deviation $\sigma'_{\bigcirc}$ for $P_{\bigcirc}$, $\langle \sigma'_{\bigcirc}, \sigma_{\square} \rangle_{v_0} = v_0 (v_1 v_2)^{\omega}$, Gain $= (1, 0)$ $\rightarrow$ not an NE;

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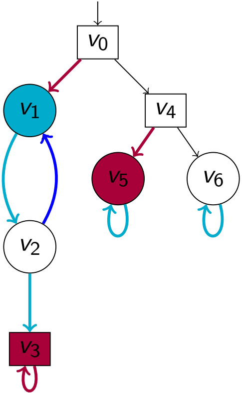

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- only one way to improve his gain;

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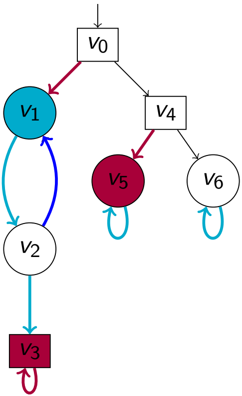

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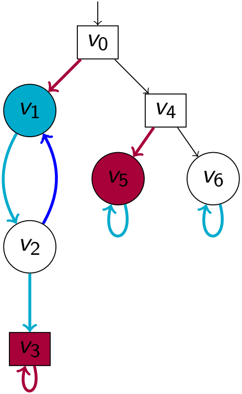

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- only one way to improve his gain;
- $\sigma'_{\bigcirc}$ not finitely deviating $\rightarrow$ weak NE.

Studied problem

Constraint problem

Let $x, y \in \{0,1\}^{|\Pi|}$ be two thresholds, decide if there exists a weak SPE $\overline{\sigma}$ in $(\mathcal{G}, v_0)$ such that $x \leq \mathsf{Gain}(\langle\overline{\sigma}\rangle_{v_0}) \leq y$.

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	exp. Muller	Muller	co-Büchi	Parity	Streett	Rabin
P-complete	$\times$					
NP-complete		$\times$	$\times$	$\times$	$\times$	$\times$

Rem : Büchi is NP-easy.

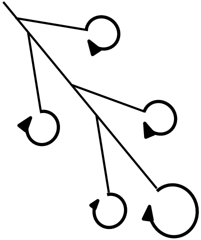
Characterization

(Good) Symbolic witness (1/2)

Symbolic witness $\mathcal{P}$

A *symbolic witness* $\mathcal{P}$ is:

- a set of lassoes with a polynomial size representation,
- there is a polynomial number of lassoes in $\mathcal{P}$,



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Good Symbolic witness $\mathcal{P}$

A **good** *symbolic witness* $\mathcal{P}$ is:

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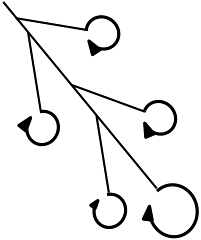
- there is a polynomial number of lassoes in $\mathcal{P}$,
- these lassoes have some "**good**" properties.



(Good) Symbolic witness (2/2)

Theorem

Let $(\mathcal{G}, v_0)$ be a Boolean game with prefix-independent gain functions. Are equivalent:

1. there exists a weak SPE in $(\mathcal{G}, v_0)$ with payoff p;
2. there exists a *symbolic witness* $\mathcal{P}$ that contains a lasso with payoff p;
3. there exists a weak SPE in $(\mathcal{G}, v_0)$ with payoff p and finite memory in $\mathcal{O}(|\Pi| \times |V|^3)$.

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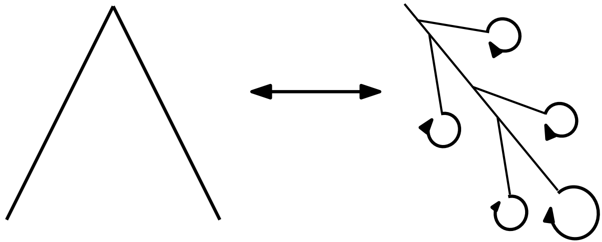


Folk theorem (1/2)

Main idea: computation of the set of all possible payoffs of a weak SPE from a given vertex v: $\mathbf{P}_{k^*}(v)$.

Folk theorem

Let $(\mathcal{G}, v_0)$ be a Boolean game with prefix-independent gain functions, there exists a weak SPE $\overline{\sigma}$ with payoff p in $(\mathcal{G}, v_0)$ if and only if $\mathbf{P}_{k^*}(v) \neq \emptyset$ for all $v \in \mathrm{Succ}^*(v_0)$ and $p \in \mathbf{P}_{k^*}(v_0)$.

Folk theorem (2/2)

How to compute the sets $\mathbf{P}_{k^*}(v)$?

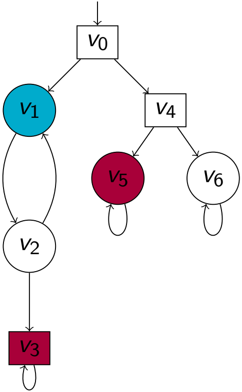
Folk theorem (2/2)

How to compute the sets $\mathbf{P}_{k^*}(v)$?

- step 0:begin with all the realizable payoffs from v,
  i.e., $p \in \mathbf{P}_0(v)$ iff $\exists \rho$ beginning in v such that $\text{Gain}(\rho) = p$;
- step k: **remove** some payoffs, from $\mathbf{P}_k(v)$, which can't be payoffs of a weak SPE and **adjust** the set $\mathbf{P}_k(v')$ of the other vertices v';
- final step: reach a fixpoint $\mathbf{P}_{k^*}(v)$.

Folk theorem (2/2)

How to compute the sets $\mathbf{P}_{k^*}(v)$?



Game with Büchi objectives:

- Player ◯: $\{v_1\}$;
- Player □: $\{v_3, v_5\}$;

	v_0	v_1	v_2	v_3	v_4	v_5	v_6
$\mathbf{P}_0$	$\{(0,0),(1,0),(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(0,1)\}$	$\{(\mathbf{0},\mathbf{0}),(0,1)\}$	$\{(0,1)\}$	$\{(0,0)\}$
$\mathbf{P}_1$	$\{\mathbf{(0,0)},(1,0),(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(0,1)\}$	$\{(0,1)\}$	$\{(0,1)\}$	$\{(0,0)\}$
$\mathbf{P}_2$	$\{\mathbf{(1,0)},(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(0,1)\}$	$\{(0,1)\}$	$\{(0,1)\}$	$\{(0,0)\}$
$\mathbf{P}_{k^*}$	$\{(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(0,1)\}$	$\{(0,1)\}$	$\{(0,1)\}$	$\{(0,0)\}$

Reachability and safety

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  - Reachability: weak SPE $\leftrightarrow$ SPE,

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  - Safety: thanks to previous results [Umm05] and the structure of our proof,

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- **SPEs:**
  - Reachability: weak SPE $\leftrightarrow$ SPE, PSPACE-complete;
  - Safety: thanks to previous results [Umm05] and the structure of our proof, PSPACE-complete.

Conclusion and future works

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weak SPEs :

	Explicit Muller	Co-Büchi	Parity	Muller	Rabin	Streett	Reachability	Safety
P-complete	$\times$							
NP-complete		$\times$	$\times$	$\times$	$\times$	$\times$		
PSPACE-complete							$\times$	$\times$

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P-complete	×							
NP-complete		×	×	×	×	×		
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- open for Büchi, NP-easyness is known;

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PSPACE-complete							$\times$	$\times$

- open for Büchi, NP-easyness is known;
- FPT (fixed parameter tractable).

Conclusion

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- Existence of weak SPEs which need “small” memory;
- Folk theorem;

Complexity results:

weak SPEs :

	Explicit Muller	Co-Büchi	Parity	Muller	Rabin	Streett	Reachability	Safety
P-complete	$\times$							
NP-complete		$\times$	$\times$	$\times$	$\times$	$\times$		
PSPACE-complete							$\times$	$\times$

- open for Büchi, NP-easyness is known;
- FPT (fixed parameter tractable).

SPEs:

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SPEs:

- Reachability and Safety : PSPACE-complete.

Future works

- Exact complexity class for Boolean games with Büchi objectives;
- Constraint problem for games with quantitative gain functions;
- Extension to SPE;
- . . .

Thomas Brihaye, Véronique Bruyère, Noémie Meunier, and Jean-François Raskin.
Weak subgame perfect equilibria and their application to quantitative reachability.
In *24th EACSL Annual Conference on Computer Science Logic, CSL 2015, September 7-10, 2015, Berlin, Germany*, pages 504–518, 2015.

Véronique Bruyère, Stéphane Le Roux, Arno Pauly, and Jean-François Raskin.
On the existence of weak subgame perfect equilibria.
In *Foundations of Software Science and Computation Structures - 20th International Conference, FOSSACS 2017, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2017, Uppsala, Sweden, April 22-29, 2017, Proceedings*, pages 145–161, 2017.

Michael Ummels.
Rational Behaviour and Strategy Construction in Infinite Multiplayer Games.
Diploma thesis, RWTH Aachen, 2005.

Classical ω-regular winning condition

A play $\rho = \rho_0\rho_1\rho_2\ldots$ satisfies one of the following winning conditions iff

- **Reachability** given $F \subseteq V$, $\mathsf{Occ}(\rho) \cap F \neq \emptyset$;
- **Safety** given $F \subseteq V$, $\mathsf{Occ}(\rho) \cap F = \emptyset$;
- **Büchi** given $F \subseteq V$, $\mathsf{Inf}(\rho) \cap F \neq \emptyset$;
- **Co-Büchi** given $F \subseteq V$, $\mathsf{Inf}(\rho) \cap F = \emptyset$;
- **Parity** $\Omega : V \to \{1, \ldots, d\}$, $\max(\mathsf{Inf}(\Omega(\rho)))$ is even;
- **Explicit Muller** given $\mathcal{F} \subseteq \mathcal{P}(V)$, $\mathsf{Inf}(\rho) \in \mathcal{F}$;
- **Muller** given a coloring function $\Omega : V \to \{1, \ldots, d\}$, and $\mathcal{F} \subseteq \mathcal{P}(\Omega(V))$, $\mathsf{Inf}(\Omega(\rho)) \in \mathcal{F}$;
- **Rabin** given $(G_j, R_j)_{1 \leq j \leq k}$ a family of pair of sets $G_j, R_j \subseteq V$, there exists $j \in 1, \ldots, k$ such that $\mathsf{Inf}(\rho) \cap G_j \neq \emptyset$ and $\mathsf{Inf}(\rho) \cap R_j = \emptyset$;
- **Streett** given $(G_j, R_j)_{1 \leq j \leq k}$ a family of pair of sets $G_j, R_j \subseteq V$, for all $j \in 1, \ldots, k$ $\mathsf{Inf}(\rho) \cap G_j = \emptyset$ or $\mathsf{Inf}(\rho) \cap R_j \neq \emptyset$.

Symbolic witness

Symbolic witness

Let $(\mathcal{G}, v_0)$ be an initialized Boolean game with prefix-independent gain functions. Let $I \subseteq (\Pi \cup \{0\}) \times V$ be the set

$$I = \{(0, v_0)\} \ \cup \ \{(i, v') \mid \text{ there exists } (v, v') \in E$$

$$\text{such that } v, v' \in \text{Succ}^*(v_0) \text{ and } v \in V_i\}.$$

A *symbolic witness* is a set $\mathcal{P} = \{\rho_{i,v} \mid (i, v) \in I\}$ such that each $\rho_{i,v} \in \mathcal{P}$ is a lasso of G with $\text{First}(\rho_{i,v}) = v$ and with length bounded by $2 \cdot |V|^2$.

A symbolic witness has thus at most $|V| \cdot |\Pi| + 1$ lassoes (by definition of I) with polynomial length.

FPT

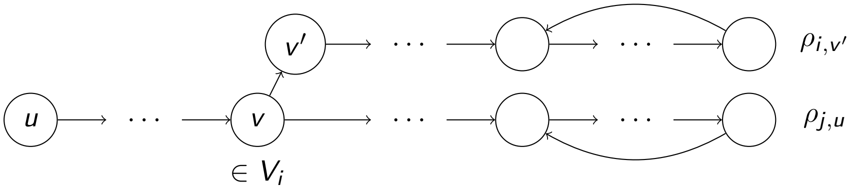
Intuitively, a language is in FPT if there is an algorithm running in polynomial time with respect to the input size times some computable function on the parameter.

Let $\mathcal{G}$ be a Boolean game. The constraint problem is in FPT for Reachability, Safety, Büchi, co-Büchi, Parity, Muller, Rabin, and Streett objectives. The parameters are

- the number $|\Pi|$ of players for Reachability, Safety, Büchi, co-Büchi, and Parity objectives,
- the number $|\Pi|$ of players and the numbers k_i, $i \in \Pi$, of pairs $(G^i_j, R^i_j)_{1 \leq j \leq k_i}$, for Rabin and Streett objectives, and
- the number $|\Pi|$ of players, the numbers d_i, $i \in \Pi$, of colors and the sizes $|\mathcal{F}_i|$, $i \in \Pi$, of the families of subsets of colors for Muller objectives.

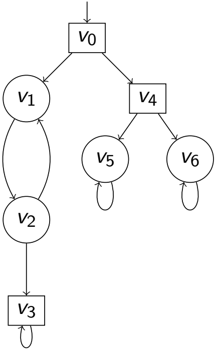
Good symbolic witness

A symbolic witness $\mathcal{P}$ is *good* if for all $\rho_{j,u},\ \rho_{i,v'} \in \mathcal{P}$, for all vertices $v \in \rho_{j,u}$ such that $v \in V_i$ and $v' \in \mathrm{Succ}(v)$, we have $\mathrm{Gain}_i(\rho_{j,u}) \geq \mathrm{Gain}_i(\rho_{i,v'})$.



Symbolic witness

Example



	$(0, v_0)$	$(2, v_4)$	$(1, v_2)$	$(1, v_1)$	$(1, v_3)$	$(2, v_5)$	$(2, v_6)$	$(1, v_5)$	$(1, v_6)$
lasso	$v_0 v_1 v_2 v_3^\omega$	$v_4 v_5^\omega$	$v_2 v_3^\omega$	$v_1 v_2 v_3^\omega$	v_3^ω	v_5^ω	v_6^ω	v_5^ω	v_6^ω
payoff	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,1)$	$(0,0)$

	v_0	v_1	v_2	v_3	v_4	v_5	v_6
$\mathbf{P}_0$	$\{(0,0),(1,0),(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(0,1)\}$	$\{(\mathbf{0},\mathbf{0}),(0,1)\}$	$\{(0,1)\}$	$\{(0,0)\}$
$\mathbf{P}_1$	$\{\mathbf{(0,0)},(1,0),(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(0,1)\}$	$\{(0,1)\}$	$\{(0,1)\}$	$\{(0,0)\}$
$\mathbf{P}_2$	$\{\mathbf{(1,0)},(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(0,1)\}$	$\{(0,1)\}$	$\{(0,1)\}$	$\{(0,0)\}$
$\mathbf{P}_{k^*}$	$\{(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(1,0),(0,1)\}$	$\{(0,1)\}$	$\{(0,1)\}$	$\{(0,1)\}$	$\{(0,0)\}$

Remove-Adjust procedure

(Remove) for all odd k: if there exists $v \in V_i$ and there exists $p \in \mathbf{P}_{k-1}(v)$ such that there exists $v' \in \mathrm{Succ}(v)$ such that for all $p' \in \mathbf{P}_{k-1}(v')$ we have: $p_i < p'_i$, then

- $\mathbf{P}_k(v) = \mathbf{P}_{k-1}(v) \setminus \{p\}$
- for all $u \neq v$ $\mathbf{P}_k(u) = \mathbf{P}_{k-1}(u)$.

[1] $\rho = \rho_0\rho_1\rho_2\dots$ is (p,k)-labeled if for all $n \in \mathbb{N}$ $p \in \mathbf{P}_k(\rho_n)$

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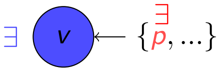

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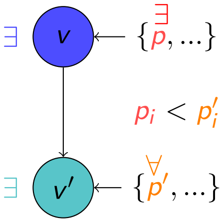

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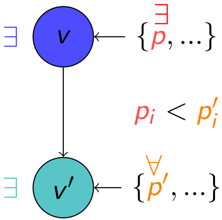

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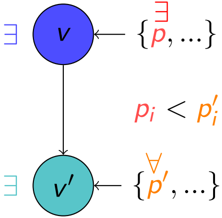

(Adjust) for all even k: let p be the payoff removed from $\mathbf{P}_{k-1}(v)$ at the Remove step for some v, we check if for all $u \in V$ such that $p \in \mathbf{P}_{k-1}(u)$ there exists a play $(p, k-1)$-labeled [1] with payoff p from u.

- yes: $\mathbf{P}_k(u) = \mathbf{P}_{k-1}(u) \setminus \{p\}$;
- no: $\mathbf{P}_k(u) = \mathbf{P}_{k-1}(u)$;
- for all u such that $p \notin \mathbf{P}_{k-1}(u)$: $\mathbf{P}_k(u) = \mathbf{P}_{k-1}(u) \setminus \{p\}$.

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