On Subgame Perfect Equilibria in Turn-Based Reachability Timed Games

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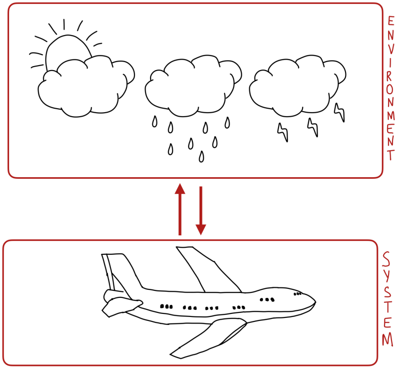
FORMATS'21

Context

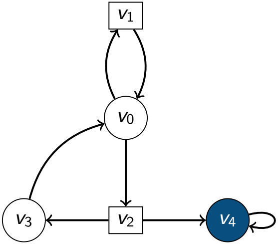
Verification and synthesis



- "Testing shows the presence, not the absence of bugs" – Edsger Dijkstra.
- **Verification: checking** that the **system** satisfies some **specifications**.
- **Synthesis: building** a system which satisfies some specifications by construction.

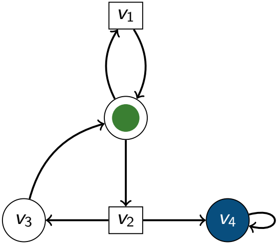
  $\hookrightarrow$ games played on graph.

Two-player zero-sum (turn-based) reachability games



- Player ◯: **the system**
- Player □: **the environment**

Two-player zero-sum (turn-based) reachability games



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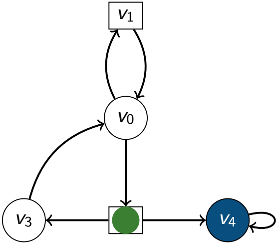

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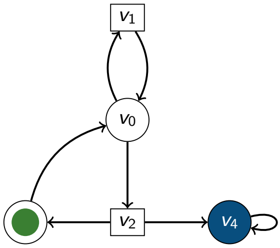

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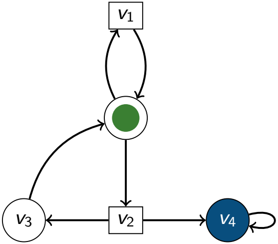

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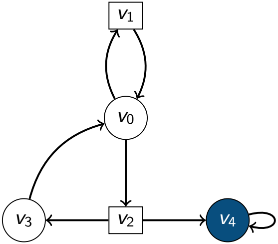
Two-player zero-sum (turn-based) reachability games



- Player ◯: **the system**
  Goal: **satisfying a property.**
  Here: reaching a vertex of the target set $F_{\bigcirc} = \{v_4\}$ (**reachability objective**)

- Player □: **the environment**
  Goal: **avoid that.**

The system satisfies the property
$\Leftrightarrow$
Player ◯ has a **winning strategy**.

Too restrictive model

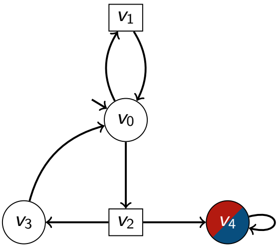
⊕ ≥ 2 players (non-antagonistic behavior) $\rightsquigarrow$ **multiplayer** games

⊕ Real time features $\rightsquigarrow$ arena of the game: **timed automaton**

Multiplayer (turn-based) Reachability Timed Games

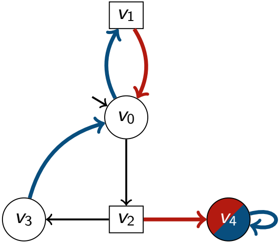
(Reachability) Multiplayer Turn-Based Games

(Turn-based) Multiplayer Reachability Games



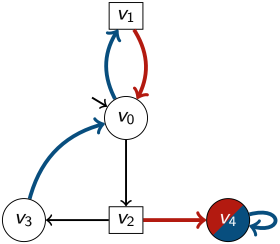
- **Two** (or more) players;
  Ex: Player ◯ and Player □.
- Objectives:
  - Player ◯ wants to reach $F_{\bigcirc} = \{v_4\}$;
  - Player □ wants to **reach** $F_{\Box} = \{v_4\}$.
  - $\rightsquigarrow$ non antagonistic.
- For every infinite path (called **play**) ρ, $\rho = \rho_0\rho_1 \ldots$,
  - $\text{Gain}_{\bigcirc}(\rho) = \begin{cases} 1 & \text{if } \exists k \text{ st. } \rho_k \in F_{\bigcirc} \\ 0 & \text{otherwise} \end{cases}$
  - $\text{Gain}_{\Box}(\rho) = \begin{cases} 1 & \text{if } \exists k \text{ st. } \rho_k \in F_{\Box} \\ 0 & \text{otherwise} \end{cases}$
- Ex:
  - $\text{Gain}((v_0v_1)^\omega) = (\text{Gain}_{\bigcirc}((v_0v_1)^\omega), \text{Gain}_{\Box}((v_0v_1)^\omega)) = (0, 0)$.
  - $\text{Gain}(v_0v_2v_3v_0v_2v_4^\omega) = (1, 1)$

(Turn-based) Multiplayer Reachability Games



- Strategy: $\sigma_i : V^* V_i \to V$;
  Ex: $\sigma_{\bigcirc}$ and $\sigma_{\Box}$
- A strategy profile: $(\sigma_{\bigcirc}, \sigma_{\Box}) \rightsquigarrow \langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_{v_0} = (v_0 v_1)^\omega$ (called **outcome**)
- ~~winning strategies~~ (optimality) $\rightsquigarrow$ other solution concepts: **equilibria** (stability).

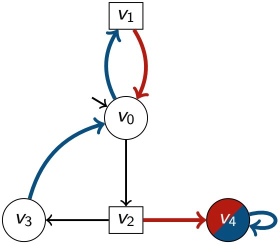
Nash equilibria



Nash equilibrium

A **strategy profile** $(\sigma_{\bigcirc}, \sigma_{\Box})$ is a Nash equilibrium (NE) if **no** player has an **incentive to deviate unilaterally**.

Nash equilibria

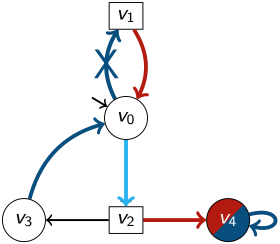


Nash equilibrium

A **strategy profile** $(\sigma_{\bigcirc}, \sigma_{\Box})$ is a Nash equilibrium (NE) if **no** player has an **incentive to deviate unilaterally**.

- Counter-ex: $(\sigma_{\bigcirc}, \sigma_{\Box})$:
  - $(\sigma_{\bigcirc}, \sigma_{\Box}) \rightsquigarrow \langle \sigma_{\bigcirc}, \sigma_{\Box} \rangle_{v_0} = (v_0 v_1)^\omega$;
  - $\text{Gain}((v_0 v_1)^\omega) = (0, 0)$.

Nash equilibria



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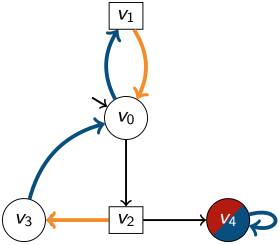
$\rightsquigarrow$ not an NE.

Nash equilibria



Nash equilibrium

A **strategy profile** $(\sigma_\bigcirc, \sigma_\square)$ is a Nash equilibrium (NE) if **no** player has an **incentive to deviate unilaterally**.

- Counter-ex: $(\sigma_\bigcirc, \sigma_\square)$:
  - $(\sigma_\bigcirc, \sigma_\square) \rightsquigarrow \langle \sigma_\bigcirc, \sigma_\square \rangle_{v_0} = (v_0 v_1)^\omega$;
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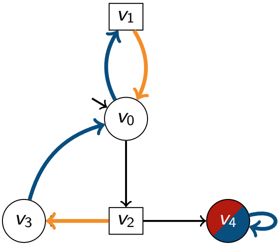
  $\rightsquigarrow$ not an NE.
- Ex: $(\sigma_\bigcirc, \sigma_\square)$:
  - $(\sigma_\bigcirc, \sigma_\square) \rightsquigarrow \langle \sigma_\bigcirc, \sigma_\square \rangle_{v_0} = (v_0 v_1)^\omega$;
  - $\text{Gain}((v_0 v_1)^\omega) = (0, 0)$.

  $\rightsquigarrow$ NE

!! Uncredible threat

Subgame Perfect Equilibria

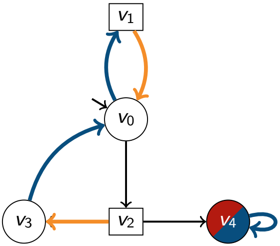


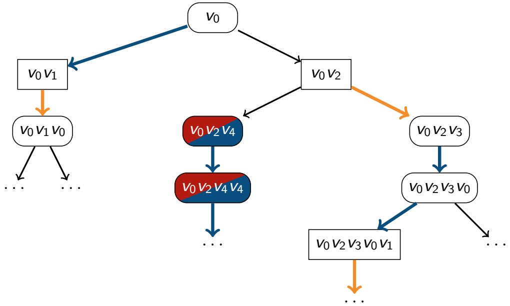
- refined solution concept: **subgame perfect equilibrium**.

Subgame perfect equilibrium

A strategy profile $(\sigma_{\bigcirc}, \sigma_{\Box})$ is a subgame perfect equilibrium (SPE) if it is an NE from each history.

Subgame Perfect Equilibria





- $(\sigma_{\bigcirc}, \sigma_{\square})$ is an **NE**;
- $(\sigma_{\bigcirc}, \sigma_{\square})$ is **not an SPE**:
  there is a **profitable deviation** from $v_0 v_2$.

Constrained existence problem

- Different equilibria may coexist !

Constrained existence problem

Given $(x_1, \ldots, x_n) \in \{0,1\}^n$ and $(y_1, \ldots, y_n) \in \{0,1\}^n$,
does **there exist** an equilibrium (NE or SPE) $(\sigma_1, \ldots, \sigma_n)$ such that for all $1 \leq i \leq n$:
$x_i \leq \text{Gain}_i(\langle \sigma_1, \ldots, \sigma_n \rangle_{v_0}) \leq y_i$?

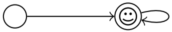
Rem: if $x_i = 1$ then Player i has to win; if $y_i = 0$ then Player i has to lose.

- **For NEs:** NP-complete [CFGR16]
- **For SPEs:** PSPACE-complete [BBGR18]

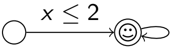
$\rightsquigarrow$ complexity gap

(Reachability) Timed Automata

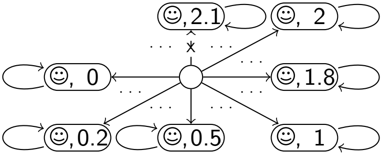
A (very) simple example



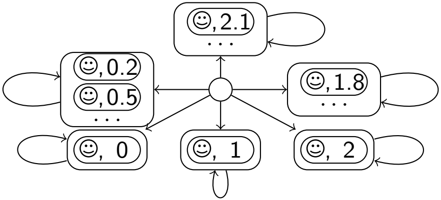
(a) Automaton



(b) Timed automaton



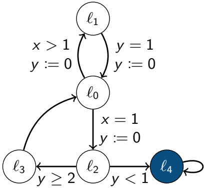
(c) Associated transition system



(d) Region graph

A (less) simple example

Timed automaton $\mathcal{A}$



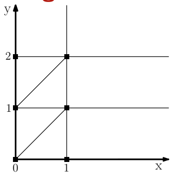
- **clocks**: x and y;
- clock **reset**: $y := 0$;
- **guards**: *e.g.*, $y \geq 2$ or $x = 1$.
  May be a finite conjunction of $c_i \diamond x_i$ where $\diamond \in \{<, \leq, =, \geq, >\}$ and x_i in $\mathbb{N}$.
- Goal $= \{\ell_4\}$

(Infinite) Transition system $T_{\mathcal{A}}$

Much more difficult to draw ...
For example, a possible path in this transition system:
$(\ell_0, (0; 0)) \xrightarrow{d=1} (\ell_2, (1; 0)) \xrightarrow{d=2.4} (\ell_3, (3.4; 2.4))$ where $d \in \mathbb{R}^+$.

$\approx$ **time-abstract bisimulation** on $T_{\mathcal{A}}$
$\rightsquigarrow$ finite number of regions
$\rightsquigarrow$ **(Finite) Region graph** (! size **exponential** in the size of $\mathcal{A}$)

Regions

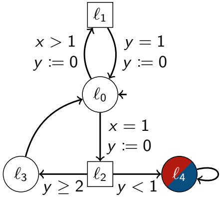


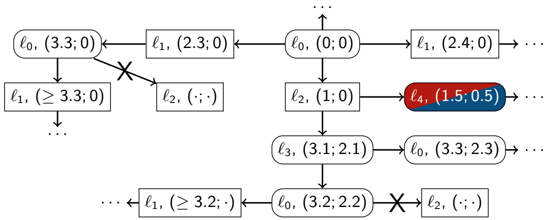
(Reachability) Timed Games

Reachability timed games

Partitioned Timed Automaton (PTA)

Semantic $\rightsquigarrow$ (Infinite) Turn-Based Reachability Game: **Reachability Timed Game**





Existence of Nash Equilibria?
Existence of Subgame Perfect Equilibria?

Studied problem

Constrained existence problem

Given a PTA $\mathcal{A}$, $(x_1, \ldots, x_n) \in \{0,1\}^n$ and $(y_1, \ldots, y_n) \in \{0,1\}^n$, does **there exist** an equilibrium (NE or SPE) $(\sigma_1, \ldots, \sigma_n)$ in its associated reachability timed game such that for all $1 \leq i \leq n$:
$x_i \leq \mathsf{Gain}_i(\langle \sigma_1, \ldots, \sigma_n \rangle_{(\ell_0,0)}) \leq y_i$?

- **For NEs**: EXPTIME-complete [Bre12];
- **For SPEs**: EXPTIME-complete [BG20] **(This talk)**.

$\rightsquigarrow$! no complexity gap

Deciding the constrained existence problem

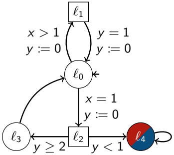
- ✓ Deciding the constrained existence problem in **finite** reachability games is PSPACE-complete.

- ✓ Deciding the constrained existence problem in **finite** reachability games is PSPACE-complete.
- ✗ The reachability timed game is **infinite**.

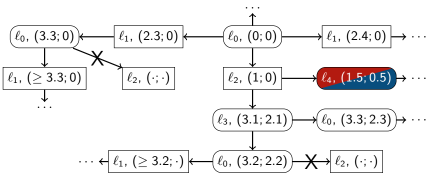
① Reachability timed game $\longrightarrow$ finite reachability game
(**Region game**)

Region game

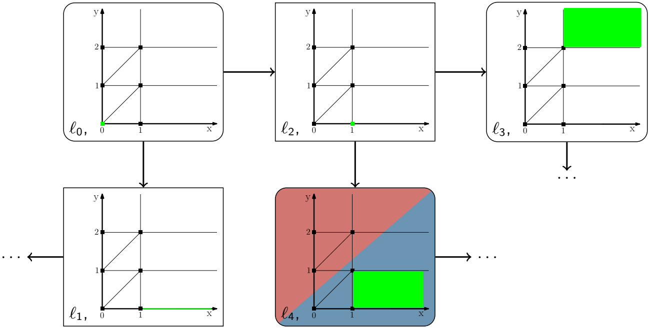
Partitioned Timed Automaton



Reachability Timed Game

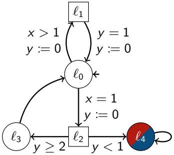


Region game $\rightsquigarrow$ **finite** reachability game !

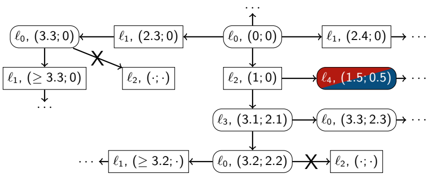


Region game

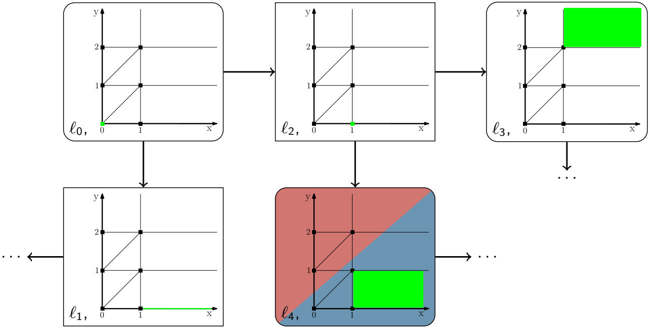
Partitioned Timed Automaton



Reachability Timed Game



Region game $\rightsquigarrow$ **finite** reachability game ! but **exponential** size



(2) $\exists$ SPE $\Leftrightarrow$ in the **reachability timed game** and in the **region game**

Theorem

Let $p \in \{0,1\}^n$,

There exists an SPE in the **Reachability Timed Game** with gain profile p

if and only if

there exists an SPE in the associated **Region Game** with gain profile p.

$\rightsquigarrow$ one can decide the constrained existence problem in the region game !

- ✓ Deciding the constrained existence problem in **finite** reachability games is PSPACE-complete.

- ✓ Deciding the constrained existence problem in **finite** reachability games is PSPACE-complete.
- ✗ The graph of the region game has a size exponential in the size of the PTA $\mathcal{A}$
  $\rightsquigarrow$ (naively) EXPSPACE algorithm.

(3) EXPTIME algorithm to decide the constrained existence problem in **finite** reachability games

③ EXPTIME algorithm to decide the constrained existence problem in **finite** reachability games

(Details not given in this talk)

- based on an SPE outcome characterization inspired by the one used in Quantitative Reachability (finite) Games [BBG+19];
- running time **exponential** in the number of players but **not** in the size of the **game graph**.

To summarize

- ✓ This latter algorithm decides the constrained existence problem in **finite** reachability game in time complexity **exponential** in the number of players but **polynomial** in the size of the graph of the game.
- ✓ The graph of the region game is **finite** and has a size exponential in the size of the underlying timed automaton of $\mathcal{A}$.
- ✓ Deciding the constrained existence problem is equivalent in the reachability timed game or in the region game.

Algorithm to decide the constrained existence problem of SPEs in reachability timed games: runs in **exponential time** (both in the **number of players** and in **the size of** the underlying timed automaton of $\mathcal{A}$)

Conclusion

Conclusion

- The constrained existence problem of SPEs in Reachability Timed Games is in EXPTIME;
- First time that SPEs are studied in this kind of setting (up to our knowledge);
- No complexity gap between NEs and SPEs;

- If there exists an SPE with a given gain profile in the Reachability Timed Game, there exists an SPE with the same gain profile in its associated Region Game.
- Given a bisimulation equivalence on a game which respects some "good" properties, we proved the same result between the game and its quotient game.
- Allows us to handle other kind of objectives?

References I

- Thomas Brihaye, Véronique Bruyère, Aline Goeminne, Jean-François Raskin, and Marie van den Bogaard, The complexity of subgame perfect equilibria in quantitative reachability games, CONCUR 2019, 2019, pp. 13:1–13:16.
- Thomas Brihaye, Véronique Bruyère, Aline Goeminne, and Jean-François Raskin, Constrained existence problem for weak subgame perfect equilibria with ω-regular boolean objectives, GandALF 2018, 2018, pp. 16–29.
- Thomas Brihaye and Aline Goeminne, On subgame perfect equilibria in turn-based reachability timed games, Formal Modeling and Analysis of Timed Systems - 18th International Conference, FORMATS 2020, Vienna, Austria, September 1-3, 2020, Proceedings (Nathalie Bertrand and Nils Jansen, eds.), Lecture Notes in Computer Science, vol. 12288, Springer, 2020, pp. 94–110.
- Romain Brenguier, Nash equilibria in concurrent games : application to timed games, Theses, École normale supérieure de Cachan - ENS Cachan, November 2012.
- Rodica Condurache, Emmanuel Filiot, Raffaella Gentilini, and Jean-François Raskin, The complexity of rational synthesis, ICALP 2016, 2016, pp. 121:1–121:15.