Permissive Equilibria in Multiplayer Reachability Games

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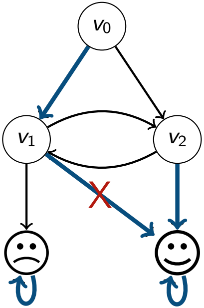
Context

Context



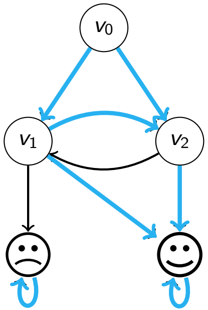
Objective: $P_{\bigcirc}$ wants to reach ☺ from v_0.

⇝ $\sigma_{\bigcirc}$ is a **winning strategy**.

What happens if the edge $(v_1, ☺)$ becomes **unavailable**?

⇝ choosing (v_1, v_2) is also winning.
⇝ strategies with multiple choices (**multi-strategies**).

Context



Objective: $P_{\bigcirc}$ wants to reach ☺ from v_0.

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What happens if the edge (v_1, ☺) becomes **un-available**?

⇝ choosing (v_1, v_2) is also winning.
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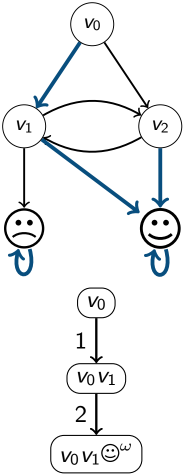
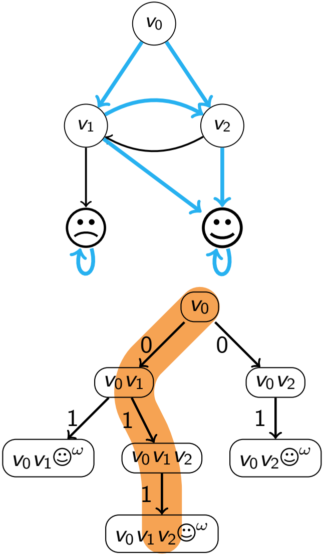
Context

How to compare two multi-strategies? Is a multi-strategy better than another one?



Penalty $= 3$



Penalty $= 2$

Context

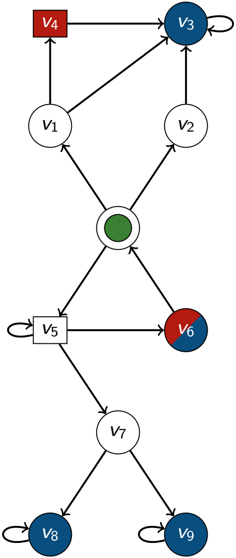
This notion of penalty is used in [BDMR09] in the setting of two-player zero-games with reachability objectives. They consider problems related to the existence of a **winning multi-strategy** with some constraint on the **penalty**.

Our goal: To extend the concept of permissiveness to the multiplayer setting. We study problems related to the existence of a **permissive equilibrium** with some constraints on the **penalties** of the players.

[BDMR09]: Measuring permissivity in finite games, P. Bouyer, M. Duflot, N. Markey and G. Renault, CONCUR'09.

Multiplayer reachability games

Reachability games

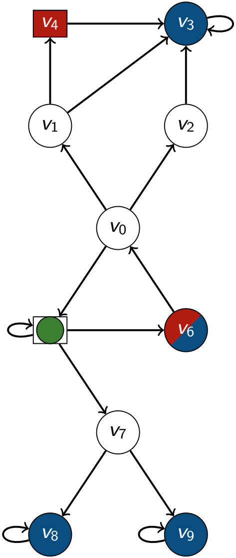


- A graph $G = (V, E)$
- A set of n players N, Ex: Player ◯ and Player □
- An initial vertex, Ex: v_0

How to play in such a game?

$$\rho = v_0$$

Reachability games



- A graph $G = (V, E)$
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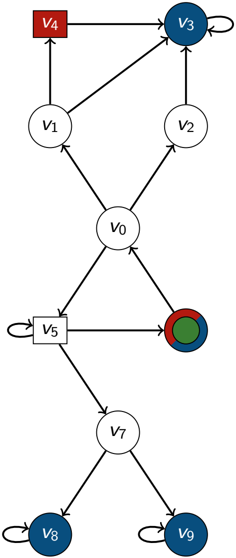
How to play in such a game?

$$\rho = v_0 v_5$$

Reachability games



- A graph $G = (V, E)$
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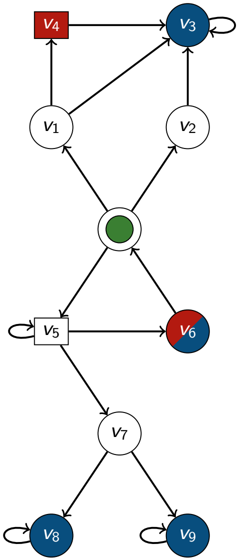
How to play in such a game?

$$\rho = v_0 v_5 v_6$$

Reachability games

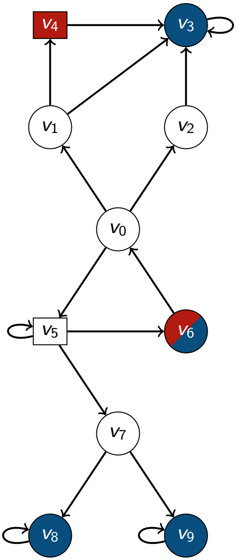


- A graph $G = (V, E)$
- A set of n players N, Ex: Player $\bigcirc$ and Player $\square$
- An initial vertex, Ex: v_0

How to play in such a game?

$$\rho = v_0 v_5 v_6 v_0 \ldots$$

Reachability games



- A graph $G = (V, E)$
- A set of n players N, Ex: Player ◯ and Player □
- An initial vertex, Ex: v_0

How to play in such a game?

$$\rho = (v_0 v_5 v_6)^\omega$$

Reachability objective

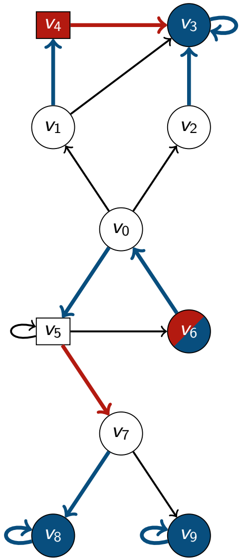
Given a target set $F_i \subseteq V$, for each play $\rho = \rho_0 \rho_1 \ldots$,

$$\text{Gain}_i(\rho) = \begin{cases} 1 & \exists k \in \mathbb{N},\ \rho_k \in F_i \\ 0 & \text{otherwise} \end{cases}$$

Ex: $F_\bigcirc = \{v_3, v_6, v_8, v_9\}$ and $F_\Box = \{v_4, v_6\}$

- $\text{Gain}((v_0 v_5 v_6)^\omega) = (\text{Gain}_\bigcirc((v_0 v_5 v_6)^\omega), \text{Gain}_\Box((v_0 v_5 v_6)^\omega)) = (1, 1)$
- $\text{Gain}(v_0 v_5 v_7 v_8^\omega) = (1, 0)$

Simple strategies and Nash equilibria

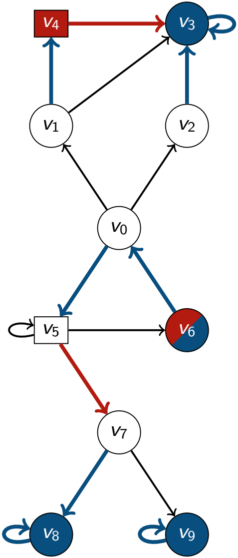


- **(Simple) strategy:** $\sigma_i : V^* V_i \to V$
  Ex: $(\sigma_{\bigcirc}, \sigma_{\square})$
- **(Simple) strategy profile:** $\sigma = (\sigma_1, \dots, \sigma_n)$
  $\rightsquigarrow \langle \sigma \rangle_{v_0}$ the **outcome**.
  Ex: $\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0} = v_0 v_5 v_7 v_8^{\omega}$.

Nash equilibrium

A **simple strategy profile** σ is a Nash equilibrium (NE) if **no** player has an incentive to **deviate** unilaterally.

Simple strategies and Nash equilibria



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Nash equilibrium

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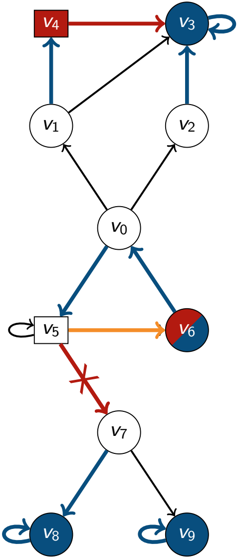
CEx:

- $(\sigma_{\bigcirc}, \sigma_{\square})$ is **not** an NE
- $\text{Gain}(\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0}) = (1, 0)$



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Nash equilibrium

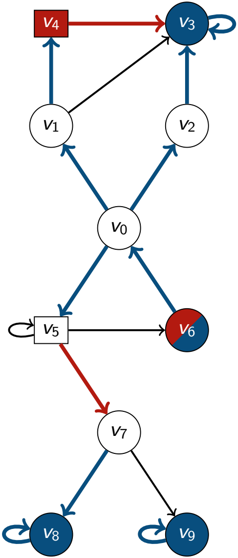
A **simple strategy profile** σ is a Nash equilibrium (NE) if **no** player has an incentive to **deviate** unilaterally.

CEx:

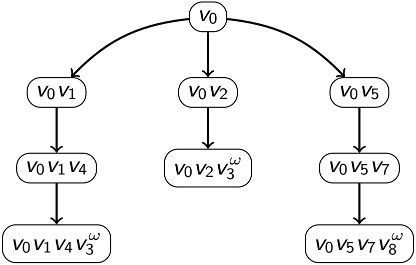
- $(\sigma_{\bigcirc}, \sigma_{\square})$ is **not** an NE
- $\text{Gain}(\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0}) = (1, 0)$
- $\sigma_{\square}$ is a profitable deviation
- $\text{Gain}_{\square}(\langle \sigma_{\bigcirc}, \sigma_{\square} \rangle_{v_0}) = \text{Gain}_{\square}((v_0 v_5 v_6)^{\omega}) = 1$

Multi-strategies and permissive Nash equilibria

Multi-strategies

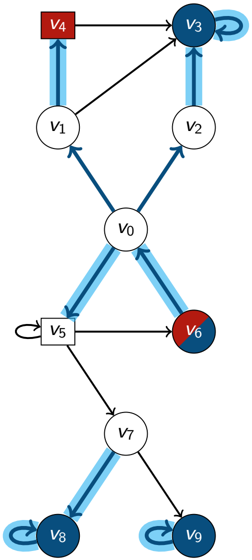


- **Multi-strategy:** $\Theta_i : V^* V_i \to \mathcal{P}(V) \setminus \{\emptyset\}$
  Ex: $(\Theta_{\bigcirc}, \Theta_{\square})$
- **Multi-strategy profile:** $\Theta = (\Theta_1, \dots, \Theta_n)$
  $\rightsquigarrow \langle \Theta \rangle_{v_0}$ the **set** of outcomes
  Ex: $\langle \Theta_{\bigcirc}, \Theta_{\square} \rangle_{v_0} = \{ v_0 v_1 v_4 v_3^{\omega}, v_0 v_2 v_3^{\omega}, v_0 v_5 v_7 v_8^{\omega} \}$



- can be seen as a **tree** $\mathcal{T}$

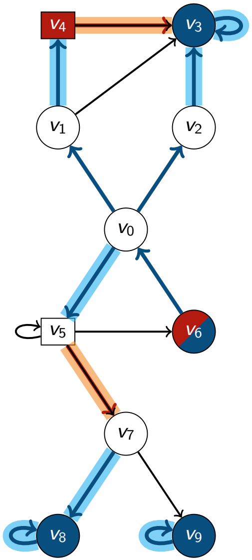
Permissive Nash equilibria



- a simple strategy σ_i is **consistent** with a multi-strategy Θ_i, $\sigma_i \lesssim \Theta_i$, if for all $hv \in V^* V_i$:

$$\sigma_i(hv) \in \Theta_i(hv).$$

Permissive Nash equilibria



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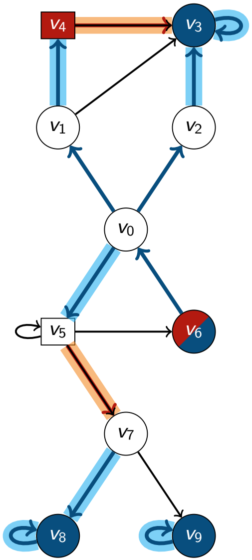
- a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ is **consistent** with a multi-strategy profile $\Theta = (\Theta_1, \ldots, \Theta_n)$ if for each $1 \leq i \leq n$, $\sigma_i \lesssim \Theta_i$.

Permissive Nash equilibria



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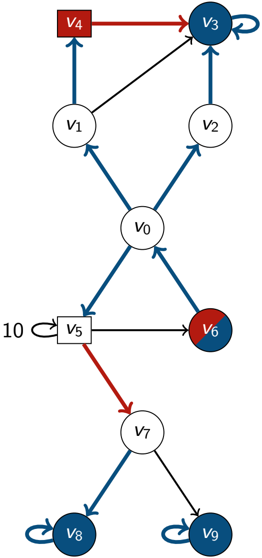
Permissive Nash equilibrium

A **multi-strategy profile** Θ is a permissive NE if each strategy profile σ consistent with Θ is an NE.

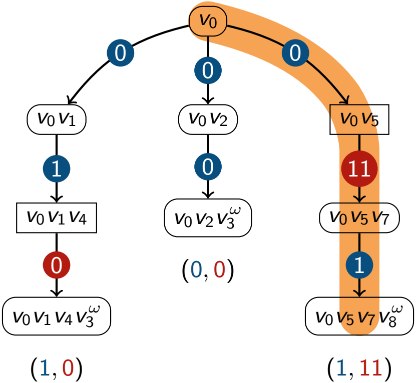
CEx:

- $(\Theta_{\bigcirc}, \Theta_{\square})$ is **not** a permissive NE;
- because $(\sigma_{\bigcirc}, \sigma_{\square})$ is **not** an NE.

Penalties



- $w : E \to \mathbb{N}$ a **weight function**



Penalties : $(1, 11)$

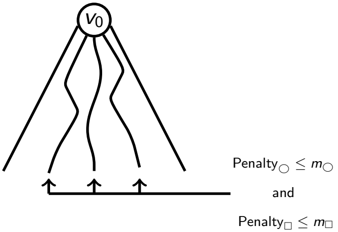
Studied problems

Constrained penalty problems

Constrained penalty problem

Given $(m_1, \ldots, m_n) \in (\mathbb{N} \cup \{+\infty\})^n$, does there exist a **permissive NE** Θ such that for each $1 \leq i \leq n$:

$$\text{Penalty}_i(\Theta) \leq m_i.$$

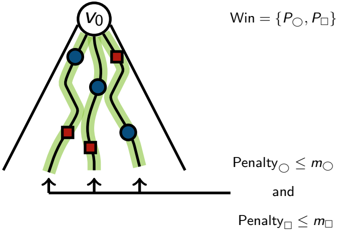


Strongly winning with constrained penalty problem

Given $(m_1, \ldots, m_n) \in (\mathbb{N} \cup \{+\infty\})^n$ and $\text{Win} \subseteq N$,
does there exist a **permissive NE** Θ such that for each $1 \leq i \leq n$:

$$\text{Penalty}_i(\Theta) \leq m_i$$

and Θ is **strongly winning w.r.t.** Win.



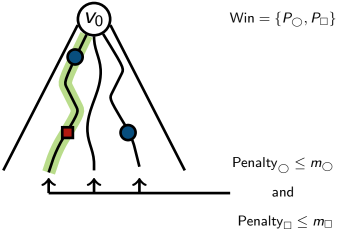
Constrained penalty problems

Weakly winning with constrained penalty problem

Given $(m_1, \ldots, m_n) \in (\mathbb{N} \cup \{+\infty\})^n$ and $\text{Win} \subseteq N$,
does there exist a **permissive NE** Θ such that for each $1 \leq i \leq n$:

$$\text{Penalty}_i(\Theta) \leq m_i$$

and Θ is **weakly winning w.r.t.** Win.



If $m_1, \ldots, m_n$ are encoded in **unary**,
the constrained penalty problems belong to PSPACE.

How to solve them?

Key idea

Characterization of Outcomes of permissive Nash equilibria

Let $\mathcal{T}$ be a tree,

there exists a permissive NE $(\Theta_1, \dots, \Theta_n)$ such that
$$\langle \Theta_1, \dots, \Theta_n \rangle_{v_0} = \mathcal{T}$$
if and only if
$\mathcal{T}$ is a good tree.

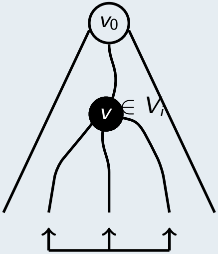
$\rightsquigarrow$ Does there exist a tree $\mathcal{T}$ such that

- each $\rho \in \mathcal{T}$ and each $i \in N$, $\text{Penalty}_i(\rho) \leq m_i$;
- $\mathcal{T}$ satisfies the property of being strongly/weakly winning;
- $\mathcal{T}$ is a good tree.

Characterization of outcomes of permissive Nash equilibria

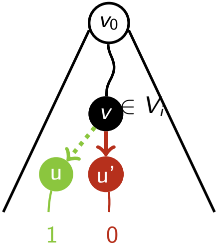
Good tree

Internal deviations

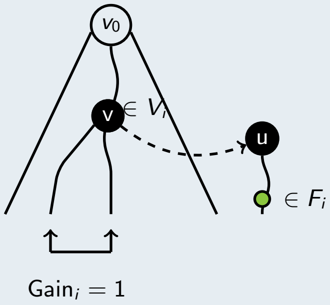


Gain$_i$ are equal

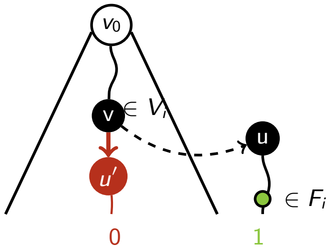
Intuition



External deviations



Intuition



Deciding the constrained penalty problems

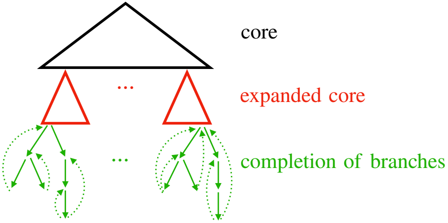
Finite symbolic tree

If there exists a tree $\mathcal{T}$ that

- satisfies the constraints given by the problem;
- is good;

then there exists a tree $\mathcal{T}'$ that

- also satisfies the constraints and is good;
- has a **finite representation**.



- This finite symbolic tree and the characterization of the outcomes of permissive NEs $\rightsquigarrow$ APTIME algorithm if thresholds are encoded in unary.

Conclusion and future works

In this work:

- permissiveness in multiplayer reachability games $\rightsquigarrow$ permissive equilibria (**Nash equilibria**, subgame perfect equilibria (SPEs))
- penalties of a multi-strategy (**main penalties**, retaliation penalties)
- decisions problems related to the existence of a permissive equilibrium with constraints on the penalties of the players
- relevant permissive equilibria
  $\rightsquigarrow$ strongly/weakly winning with constrained penalty problems
- those problems belong to PSPACE if the thresholds are encoded in unary

Patricia Bouyer, Marie Duflot, Nicolas Markey, and Gabriel Renault.
Measuring permissivity in finite games.
In Mario Bravetti and Gianluigi Zavattaro, editors, CONCUR 2009, volume 5710 of LNCS, pages 196–210. Springer, 2009.